PROPERTIES OF TWISTED SEMIGROUP RINGS

By

ANTONIO QUESADA RETTSCHLAG

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1978

ACKNOWLEDGEMENTS

I would like to express my gratitude to all the people that in some way have contributed to make this work possible; they are:

Dr. Jorge Martinez, Dr. Gerhard Ritter, Dr. Sergio Zarantonello and Dr. Elroy Bolduc, Jr., who served as members of my committee;

Dr. Bruce Edwards whose help in the computer work was invaluable;

Dr. Bolling Farmer and Dr. Robert Long who assisted in improving my background in related areas;

Dr. Mark Teply, who taught me most of the algebra that I know, and who as an advisor was always ready to listen with his limitless patience and unswerving interest. I consider it a privilege to have had the opportunity to work under his guidance.

In addition I would like to recognize the efforts and sacrifices of my family throughout my years of study. I am indebted to my parents for providing me the opportunity to complete my graduate studies in Spain and to my wife and daughter for having the patience and understanding which made the burden of my doctoral studies considerably lighter. Without the faith and love of these four people I could never have finished. To them I dedicate this work.

Finally, my thanks to Sharon for her fine and fast job of typing.

This work was performed during a sabbatical granted to me by the Catholic University of Puerto Rico.

TABLE OF CONTENTS

Page

ACKNOWLEDGEMENTS.................................ii

ABSTRACT... v

CHAPTER

I - PRELIMINARIES AND GENERAL PROPERTIES OF SEMIGROUP RINGS.. 1

1.1 Semigroups.................................. 1
1.2 Rings....................................... 4
1.3 Semigroup Rings............................. 8
1.4 Twisted Group Rings......................... 9
1.5 Twisted Semigroup Rings.....................21

II - ON NONTRIVIAL UNITS OF TWISTED GROUP RINGS AND TWISTED SEMIGROUP RINGS.............................24

2.1 Nontrivial Units in Twisted Group Rings.......26
2.2 Nontrivial Units in Twisted Semigroup Rings...32
2.3 Nontrivial Units in Twisted Semigroup Rings Over Nontorsion Free Semigroups.............35
2.4 Zero Divisors in Twisted Group Rings..........38

III - NILPOTENT RADICALS OF NONCOMMUTATIVE SEMIGROUP RINGS...41

3.1 An Example....................................42
3.2 Semigroup Rings Over Right (Left) Zero Semigroups and Their Nilradicals............46
3.3 The Nilpotent Radical of the Semigroup Ring of a Right (Left) Group....................50
3.4 Semigroup Rings Over Semigroups with Universally Minimal Ideals, and Their Nilradicals.60

IV - SEMIPRIME SEMIGROUP RINGS.........................67

4.1 Semiprime Twisted Semigroup Rings.............69
4.2 On Semigroup Rings Over Semigroups Admitting Relative Inverses..........................75

BIBLIOGRAPHY.....................................89

BIOGRAPHICAL SKETCH..............................91

Abstract of Dissertation Presented to the Graduate
Council of the University of Florida in Partial Fulfillment
of the Requirements for the Degree of Doctor of Philosophy

PROPERTIES OF TWISTED SEMIGROUP RINGS

By

Antonio Quesada Rettschlag

August 1978

Chairman: Mark L. Teply
Major Department: Mathematics

Let R be a ring with identity, and let S be a multiplicative semigroup with identity i. Let γ be a function from $S \times S$ into the group of central units of R such that $\gamma(s_1,s_2)\gamma(s_1s_2,s_3) = \gamma(s_2,s_3)\gamma(s_1,s_2s_3)$ for all $s_1,s_2,s_3 \in S$. We define the twisted semigroup ring, denoted by $R^t[S]$, as an R-algebra with basis $\{\bar{s} | s \in S\}$ and with multiplication defined distributively such that for all $s_1,s_2 \in S$,

$$\bar{s}_1\bar{s}_2 = \gamma(s_1,s_2)\overline{s_1s_2}.$$

Assuming that γ is nonconstant, we prove the following statements.

(A) If K is a field and G is a nontorsion free group, then $K^t[G]$ has nontrivial units.

(B) If S is a nontorsion free semigroup and char R is finite, then $R^t[S]$ has nontrivial units.

(C) If S is a nontorsion free semigroup and K is a field then $K^t[S]$ has nontrivial units.

(D) If $R^t[S]$ has no nontrivial units, then R has no nilpotent elements and either R is indecomposable or the largest subgroup of S containing i is trivial.

Next we let $N[R[S]]$ denote the nilpotent radical of $R[S]$. We prove the following statements.

(E) If R is a commutative ring with 1 and S is a right (left) zero semigroup, then $N[R[S]] = \{\Sigma a_i x_i \in R[S] | \Sigma a_i = 0\}$.

(F) If R is a ring with 1 and S is a right (left) group, then $N[R[S]] = N[R[H_1]] + C$, where H_1 is any maximal subgroup of S and $C = R[H_1] \cdot W$ with $W = \{ \sum_{i=1}^{n} a_i e_i | \sum_{i=1}^{n} a_i = 0$, e_i is an idempotent in S and $n \in Z^+\}$.

(G) Let R be a ring with 1, and let S be a semigroup with a universally minimal ideal U. Let θ be the semigroup homomorphism from S onto U defined by $\theta(s) = s \cdot 1_U$. Then $N[R[S]] = N[R[U]] \oplus N[K]$, where $K = \text{Ker } \bar{\theta}$ and $\bar{\theta}: R[S] \to R[U]$ is the natural extension of θ.

(H) If R is a ring and S is a semigroup satisfying Bauaschewski's condition on extremities, then $R^t[S]$ is semiprime if and only if R is semiprime.

We say that a semilattice is of type P_0 if it has a zero element w and all the nonzero idempotents are primitive. A semilattice is said to be of type P_1 if it has a zero element and satisfies the following condition: if F is a subset of P_1, then either g.l.b. $\{x | x \in F\}$ belongs to F or else there exist x and y in F such that g.l.b. $\{x,y\}$ is in $P_1 - F$.

Let S be a semigroup such that (i) S admits relative inverses and (ii) the idempotents of S commute. If the semilattice of subgroups S_α of S is of type P_0, then we

decompose R[S] as a direct sum of ideals which are completely determined once $R[S_\alpha]$ is known for each $\alpha \in P_0$. This allows us to characterize the nilpotent and Jacobson radicals of R[S]. For R field, we give necessary conditions for R[S] to be semisimple and regular.

If the semilattice of subgroups S_α of S is of type P_1, then we prove that R[S] is semiprime if $R[S_\alpha]$ is semiprime for each $\alpha \in P_1$.

CHAPTER I
PRELIMINARIES AND GENERAL PROPERTIES OF SEMIGROUP RINGS

In this chapter we introduce some fundamental defnitions and general results that will be used throughout this dissertation.

1.1 Semigroups

A semigroup S is a nonempty set closed under a binary, associative operation that we will denote multiplicatively. Let S be a semigroup. The cardinal number $|S|$ of the set S is called the order of S. If $|S|$ is finite, we can exhibit the binary operation in S by means of its Cayley multiplication table as for finite groups. An element t of S is said to be left (right) cancellable if, for any x and y in S, $tx = ty$ $(xt = yt)$ implies $x = y$. S is called left (right) cancellative if every element of S is left (right) cancellable. We say that S is cancellative if it is both left and right cancellative. An element u of S is called a left (right) identity of S if $us = s$ $(su = s)$ for all $s \in S$. An element u of S is a two-sided identity, or simply identity, if it is both a left and a right identity of S. Exactly one of the following statements must hold for a semigroup S:

(1) S has no left and no right identities;

(2) S has one or more left (right) identity, but no right (left) identity;

(3) S has a unique two-sided identity, that we denote by i, and no other right or left identity.

An element z of S is called a left (right) zero if $zs = z$ ($sz = z$) for every s in S. An element z is called a zero element of S if it is both a left and a right zero of S. Clearly the foregoing trichotomy holds if we replace the word identity by zero. S is said to be a right zero semigroup if $xy = y$ for every x,y in S; that is, every element of S is a right zero and a left identity. Left zero semigroups are defined dually. A semigroup S with a zero element z is called a zero or null semigroup if $xy = z$ for all x,y in S.

An element e of S is called idempotent if $e^2 = e$. One-sided identity and zero elements are idempotent. S is called a band if all its elements are idempotent. By a subgroup of a semigroup S, we mean a subsemigroup T of S that is also a group with respect to its binary operation; that is, $xT = Tx = T$ for every x in T. By a left (right) ideal of S we mean a nonempty subset X of S such that $SX \subseteq X$ ($XS \subseteq X$). By two-sided ideal, or simply ideal, we mean a subset of S which is both a left and a right ideal of S. A semigroup S is called left (right) simple if S itself is the only left (right) ideal of S. Likewise S is called simple if it contains no proper ideal. It follows from the definition that S is right (left) simple if and only if $aS = S$ ($Sa = S$) for every a in S; moreover, S is a group if and only if it is both a left and a right simple semigroup.

If s is an element of S, then the cyclic subsemigroup generated by s, is the set $\langle s \rangle$ of all the positive powers of S.

Theorem 1.1. Let $s \in S$. If $|\langle s\rangle|$ is infinite all the powers of s are distinct. If $|\langle s\rangle|$ is finite, there exist two positive integers, the index r and the period m of s, such that $s^{m+r} = s^r$ and $\langle s\rangle = \{s,s^2,\ldots,s^{m+r-1}\}$. The order of $\langle s\rangle$ is m+r-1. The set $K_s = \{s^r,s^{r+1},\ldots,s^{m+r-1}\}$ is a cyclic subgroup of S of order m.

An element $s \in S$ has finite order if $|\langle s\rangle|$ is finite. S is called torsion if every element of S has finite order. If s is an element of finite order in S, then $\langle s\rangle$ contains exactly one idempotent, namely the identity of K_s.

Let the semigroup S have an identity i. If x and y are elements of S such that xy = i, then x is called a left inverse of y, and y is called a right inverse of x. A right (left) unit in S is defined to be an element of S having a right (left) inverse in S. By a unit in S we mean an element of S having both right and left inverse in S.

Theorem 1.2. Let S be a semigroup with identity i.

(1) The set P(Q) of all right (left) units of S is a right (left) cancellative subsemigroup of S containing i.

(2) The set G of all units of S is a subgroup of S, and $G = P \cap Q$. Each unit has a unique two-sided inverse in G, and has no other left or right inverse in S.

(3) Every subgroup of S containing i is contained in G.

If e is an idempotent element of S, then eSe is the set of all elements of S for which e is a two-sided identity.

Theorem 1.3. Let e be an idempotent of S, and let H_e be the group of units of eSe. Then H_e contains every subgroup of S that meets H_e.

It follows that the groups H_e are just maximal subgroups of S; that is, no H_e is properly contained in any other subgroup of S. Moreover, if e and f are idempotent elements of S, $e \neq f$, then H_e and H_f are disjoint.

The preceeding terminology and results can be found in [5], where the reader can find considerable additional material on semigroups.

1.2 Rings

Let R denote an associative ring. If there is a least positive integer n such that $na = 0$ for all $a \in R$, then R is said to have characteristic n. If no such n exists, R is said to have characteristic 0. We will denote characteristic of R by char R. A ring of characteristic p^n, for a prime p, is called a p-ring [24]. (Note that there are other conflicting uses of the term p-ring; for example, see [16]). For a ring R and a prime integer p, define $R_p = \{r \in R | p^n r = 0$ for some positive integer $n\}$. The center of R is the set $C = \{c \in R | cr = rc \text{ for all } r \in R\}$.

If R has identity 1, the subring of R generated by 1 will be called the prime subring. It is an easy consequence of the definition that if π is the prime subring of a ring R, then $\pi \cong Z_n$ whenever char $R = n > 0$, and $\pi \cong Z$ otherwise.

Theorem 1.4 [13]. Let $A_1, A_2, \ldots, A_n$ be ideals in a ring R such that (i) $A_1 + A_2 + \ldots + A_n = R$ and (ii) for each k $(1 \le k \le n)$, $A_k \cap (A_1 + \ldots + A_{k-1} + A_{k+1} + \ldots + A_n) = 0$. Then there is an isomorphism $R \cong A_1 \oplus A_2 \oplus \ldots \oplus A_n$.

Lemma 1.5. Let R be a ring with nonzero characteristic n, and let $\{p_1, p_2, \ldots, p_k\}$ be the set of distinct prime divisors of n. Then $R = R_{p_1} \oplus R_{p_2} \oplus \ldots \oplus R_{p_n}$.

A nonzero element a in a ring R is said to be a left (right) zero divisor if there exists a nonzero element $b \in R$ such that $ab = 0$ $(ba = 0)$. A proper zero divisor, or simply zero divisor, is an element of R which is both a left and a right zero divisor. A commutative ring with identity $1 \neq 0$ and no zero divisors is called an integral domain.

Let a be an element of R. The principal ideal generated by a is the set $(a) = \{ra + as + na + \sum_{i=1}^{n} r_i a s_i \mid r, s, r_i, s_i \in R$ and $n \in Z\}$. An ideal P in R is said to be prime if $P \neq R$ and, for any ideals A,B in R, $AB \subseteq P$ implies that $A \subset P$ or $B \subset P$. An ideal M in R is said to be maximal if $M \neq R$ and, for every ideal A such that $M \subset A \subset R$, either $M = A$ or $A = R$.

Definition 1.6 [9]. Let *L* be a certain property that a ring may possess. We shall say that the ring R is an *L*-ring if it possesses the property *L*. An ideal I of R will be called an *L*-ideal if I is an *L*-ring. A ring which does not contain any nonzero *L*-ideals will be called *L*-semisimple. We shall call *L* a radical property if the following three conditions hold.

(A) A homomorphic image of an L-ring is an L-ring.

(B) Every ring contains an L-ideal A which contains every other L-ideal of the ring.

(C) The factor ring R/A is L-semisimple.

An element x of R is said to be nilpotent if $x^n = 0$ for some positive integer n. A ring R (an ideal A) is said to be nil if every element x of R (of A) is nilpotent. An ideal is called nilpotent if some power of it is zero. We say that n is the index of nilpotency of a nilpotent ideal A, if n is the smallest positive integer such that $A^n = 0$.

Lemma 1.7 [9]. The sum of any finite number of nilpotent ideals of a ring R is again a nilpotent ideal. The union of all the nilpotent ideals of a ring R is a nil ideal.

In [9] we find the following construction. Denote by N_0 the union of all the nilpotent ideals of R. Then R/N_0 may have nilpotent ideals. Let N_1 be the ideal of R such that N_1/N_0 is the union of all the nilpotent ideals of R/N_0. In general, for every ordinal α that is not a limit ordinal, we define N_α to be the ideal of R such that $N_\alpha/N_{\alpha-1}$ is the union of all the nilpotent ideals of $R/N_{\alpha-1}$. If α is a limit ordinal, we define $N_\alpha = \cup_{\beta<\alpha} N_\beta$. In this way we obtain an ascending chain of ideals $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots$. Since R is a set, we may then consider the smallest ordinal τ such that $N_\tau = N_{\tau+1} = \cdots$. We denote N_τ by N[R], and N[R] is called the nilpotent radical of R. ($N[R]$ is also called the Baer lower radical of R in the literature [9]). It is characterized by the fact that $R/N[R]$ is semiprime and $N[R]$ is the smallest ideal of R that gives such a factor ring.

Lemma 1.8. Let $R = \Sigma^{w}_{\lambda\in\Lambda} R_\lambda$ be the weak direct sum of the family $\{R_\lambda\}_{\lambda\in\Lambda}$ of rings. Then the nilpotent radical of R is $\Sigma^{w}_{\lambda\in\Lambda} N[R_\lambda]$.

We define the prime radical of R as the intersection of all prime ideals of R, and we denote it by $P[R]$.

Propositon 1.9 [15]. The following conditions concerning the ring R are equivalent.

(i) 0 is the only nilpotent ideal of R.

(ii) 0 is an intersection of prime ideals; that is $P[R] = 0$.

(iii) For any ideals A and B of R, $AB = 0$ implies that $A \cap B = 0$. Under these conditions R is called semiprime.

The following definitions and results can be found in [17].

Let $a \in R$. If there exists an element b of R such that $a + b - ab = 0$, a is said to be right quasi-regular (r.q.r.). The Jacobson radical $J[R]$ of a ring R is defined as follows:

$$J[R] = \{a \in R \mid ar \text{ is r.q.r. for every } r \in R\}.$$

Theorem 1.10. Let R be a ring with identity, and let $\{A_i \mid i \in I\}$ be the set of all the maximal right ideals in R. Then $J[R] = \bigcap_{i\in I} A_i$.

A ring R is called Artinian if the descending chain condition for right ideals holds in R; that is, every strictly decreasing chain of right ideals in R contains only a finite number of right ideals.

Lemma 1.11. If R is artinian, then the Jacobson radical of R coincides with the prime radical of R.

1.3 Semigroup Rings

Let R be a ring, and let S be a multiplicative semigroup. N. Jacobson [14] defines the semigroup ring of S over R to be the set of functions from S into R that are finitely nonzero, with addition and mulitplication defined as follows:

$$(f+g)(s) = f(s) + g(s)$$

$$(fg)(s) = \sum_{xy=s} f(x)g(y),$$

where the symbol $\sum_{xy=s}$ indicates that the sum is taken over all ordered pairs (x,y) of elements of S such that $xy = s$. We use R[S] to denote the semigroup ring of S over R. Semigroup rings are generalizations of polynomial rings: if Z_0 denotes the additive semigroup of nonnegative integers, then $R[Z_0]$ is isomorphic to the polynomial ring in one indeterminate over R.

We think of the elements of R[S] as "polynomials" $\sum_{i=1}^{n} a_i s_i$, where $a_i \in R$ and $s_i \in S$. A typical nonzero element α of R[S] has a unique representation $\alpha = \sum_{i=1}^{n} a_i s_i$, where the a_i are nonzero elements of R and the s_i are distinct elements of S. The unique representation $\sum_{i=1}^{n} a_i s_i$ of α is called the canonical form of α, and the set $\{s_i\}_{i=1}^{n}$ of "variables" of the canonical form of α is called the support of α (denoted by supp α); moreover, the ideal of R generated by $\{a_i\}_{i=1}^{n}$ is called the content of α.

If the semigroup S has an identity i, then the ring R is naturally embedded in the semigroup ring R[S] by mapping an element $r \in R$ to ri.

1.4 Twisted Group Rings

Let K be a field, and let G be a multiplicative group whose identity we denote by i. Passman [22] defines a twisted group ring $K^t[G]$ of G over K as an associative K-algebra with basis $\{\bar{x} \mid x \in G\}$ and with multiplication defined distributively such that, for all $x,y \in G$,

(1) $\bar{x}\,\bar{y} = \gamma(x,y)\overline{xy}, \quad \gamma(x,y) \in K - \{0\}.$

The associativity condition is clearly equivalent to $(\bar{x}\bar{y})\bar{z} = \bar{x}(\bar{y}\bar{z})$ for $x,y,z \in G$, and this is in turn equivalent to

(2) $\gamma(x,y)\gamma(xy,z) = \gamma(y,z)\gamma(x,yz)$

Since

$$(\bar{x}\bar{y})\bar{z} = \gamma(x,y)\overline{xy}\,\bar{z} = \gamma(x,y)\gamma(xy,z)\overline{xyz}$$

and

$$\bar{x}(\bar{y}\bar{z}) = \gamma(y,z)\bar{x}\,\overline{yz} = \gamma(y,z)\gamma(x,yz)\overline{xyz}\ .$$

We call $\gamma: G \times G \to K - \{0\}$ the twist function. This function has also been called factor set in representation theory; see for instance Curtis and Reiner [8].

There are a number of obvious similarities between twisted group rings and ordinary ones, but there are also a number of important differences. Passman points out the following. First, for fixed K and G there are many different possibilities for $K^t[G]$, depending on the choice of the twist function. Indeed, one such choice is always $K[G]$, which is obtained by taking $\gamma(x,y) = 1$ for all x,y. On the other hand, of course, $K[G]$ is uniquely determined by K and G. Second, G is not embedded naturally in $K^t[G]$; that is, the map $G \to K^t[G]$ given by $x \to \bar{x}$ is not a multiplicative homomorphism in view of equation (1). Third, if H is a subgroup of G, then the K-linear span of $\{\bar{h} | h \in H\}$ is clearly a twisted group ring of H over K, but we must be careful and note that this $K^t[H]$ has as a twisting the restriction of the function γ to $H \times H$. Finally, if H is a normal subgroup of G, there is no obvious homomorphism from $K^t[G]$ to some twisted group ring of G/H.

Lemma 1.12. The following relations hold in $K^t[G]$.

(i) $\gamma(x,i) = \gamma(i,z) = \gamma(i,i)$ for all $x,z \in G$.

(ii) $\gamma(i,i)^{-1}\bar{i}$ is the identity element of $K^t[G]$.

(iii) For all $x \in G$,

$$\bar{x}^{-1} = \gamma(x,x^{-1})^{-1}\gamma(i,i)^{-1}\overline{x^{-1}} = \gamma(x^{-1},x)^{-1}\gamma(i,i)^{-1}\overline{x^{-1}}.$$

Thus we see that the elements $\bar{x} \in K^t[G]$ are all units.

Proof. (i) Putting $y = i$ in equation (2) yields $\gamma(x,i) = \gamma(i,z)$. Thus for all $x,z \in G$ we have $\gamma(x,i) = \gamma(i,z) = \gamma(i,i)$.

(ii) By (i) and equation (1) we see that $\gamma(i,i)^{-1}\bar{i}$ is the identity element of $K^t[G]$.

(iii) By (i), (ii) and (1) $\gamma(x^{-1},x)^{-1}\gamma(i,i)^{-1}\overline{x^{-1}}$ is a left inverse for $\bar{x}$, and $\gamma(x,x^{-1})^{-1}\gamma(i,i)^{-1}\overline{x^{-1}}$ is a right inverse for $\bar{x}$. Thus $\bar{x}$ is invertible and $\bar{x}^{-1}$ is equal to both of these expressions.

It is quite possible that some given $K^t[G]$ is just $K[G]$ disguised in a rather natural way. We start with $K^t[G]$ and make a diagonal change of basis by replacing each $\bar{x}$ by $x' = \delta(x)\bar{x}$ for some nonzero $\delta(x) \in K$. Observe that the word "diagonal" arises here because, for G finite, the corresponding change of basis matrix would be diagonal with the δ's as diagonal entries. From (1) and the foregoing we have

$$x'y' = \delta(x)\bar{x}\delta(y)\bar{y} = \delta(x)\delta(y)\gamma(x,y)\overline{xy}$$

$$= \delta(x)\delta(y)\delta(xy)^{-1}\gamma(x,y)(xy)'$$

In other words, with this change of basis, $K^t[G]$ is realized in a second way as a twisted group ring of G over K, this time with twist function:

(3) $\gamma'(x,y) = \delta(x)\delta(y)\delta(xy)^{-1}\gamma(x,y)$.

Thus we see that if $K^{t_1}[G]$ and $K^{t_2}[G]$ are two twisted group rings of G over K and if their corresponding twist functions

γ and γ' are related as in (3) for some $\delta:G \to K - \{0\}$, then these rings are actually identical, and we say that they are diagonally equivalent. In particular, $K^t[G]$ is diagonally equivalent to $K[G]$ if and only if

$$\gamma(x,y) = \delta(xy)\delta(x)^{-1}\delta(y)^{-1}$$

for some function $\delta:G \to K - \{0\}$.

Let $K^t[G]$ be given with twist function γ. Consider $K^{t'}[G]$ with twist function γ' defined by $\gamma'(x,y) = \gamma(i,i)^{-1}\gamma(x,y)$. Clearly γ' and γ satisfy equation (3); hence $K^t[G]$ and $K^{t'}[G]$ are diagonally equivalent. Moreover by Lemma 1.12 (i) and (ii)

(4) $\bar{i}$ is the identity of $K^{t'}[G]$ and $\gamma'(x,i) = \gamma'(i,z) = \gamma(i,i)=1$

for all $x,z \in G$. In view of this, we give the following definition: a twist function is said to be normalized if it satisfies condition (4).

From now on all the twist functions considered are supposed to be normalized.

Let us consider now some classical examples. Let G be a cyclic group of order n generated by g, and let $\phi:K[x] \to K^t[G]$ be defined by $\phi(x) = \bar{g}$. Then ϕ is a ring homomorphism such that $\phi(x^n) = ai$, where $a = \gamma(g,g)\gamma(g,g^2)\ldots\gamma(g,g^{n-1})$. Thus $\phi(x^n) = \phi(a)$, and hence the kernel of ϕ is the principal ideal generated by the polynomial $x^n - a$. It follows that $K^t[G] \simeq K[x]/(x^n-a)$. Conversely, for any $a \in K - \{0\}$, we see that $K[x]/(x^n-a)$ has as a K-basis the images of $1, x, x^2, \ldots, x^{n-1}$; so it is a twisted group ring of C_n, the cyclic group of order n. Thus, for instance, if $K = Q$ is the field of rationals, then by the Eisenstein's criteria the polynomial $x^n - 2$ is always irreducible. Hence for $a = 2$ we see that $Q^t[C_n] \cong Q[x]/(x^n-2)$ is, in fact, a field extension of Q, namely $Q[\sqrt[n]{2}]$. Moreover, if K is a field of characteristic $p > 0$ that is not perfect, choose $a \in K$ with $\sqrt[p^n]{a} \notin K$. Then for all n, $x^{p^n} - a$ is irreducible over K and the field $K[x]/(x^{p^n}-a) = K[\sqrt[n]{a}]$ is a twisted group ring of the cyclic group C_{p^n}. A last application of the foregoing example is obtained when we let K be an algebraically closed field and $G = C_n$. Then $\bar{x}^n = ai$ and $\sqrt[n]{a}$ exists in K. Since $(a^{-1/n}x)^n = \bar{i}$ it follows now that $K^t[G]$ is diagonally equivalent to $K[G]$.

In the case where $G = \langle g \rangle$ is infinite cyclic, then we can make a diagonal change of basis by replacing $\overline{g^m}$ by $\bar{g}^m$ for all integers m. This clearly implies that $K^t[G]$ is diagonally equivalent to $K[G]$.

Finally, if we let $G = \{i, g_1, g_2, g_3\}$ be the Klein group, and let K be a subfield of the real numbers, then by defining

$$\gamma(g_1,g_2) = \gamma(g_2,g_3) = \gamma(g_3,g_1) = 1$$

and

$$\gamma(g_2,g_1) = \gamma(g_3,g_2) = \gamma(g_1,g_3) = \gamma(g_1,g_1) = \gamma(g_2,g_2) =$$
$$= \gamma(g_3,g_3) = -1,$$

we obtain a twisted group ring that is easily recognized as the quaternion division algebra.

Just to mention how the twisted group rings might arise we give the next two lemmas, whose proofs can be found in [22].

Lemma 1.13. Let Z be a central subgroup of G, and let L be a maximal ideal in $K[Z]$. If $I = L\cdot K[G]$, then $K[G]/I = F^t[G/Z]$ is a twisted group ring of G/Z over the field $F = K[Z]/I$.

It turns out that every twisted group ring arises in this manner as we see next.

Lemma 1.14. Let $K^t[H]$ be a twisted group ring of H over K. Then there exists a group G with a central subgroup Z such that $G/Z \cong H$ and $K[G]/I \simeq K^t[H]$, where $I = L\cdot K[G]$ and L is an ideal of $K[Z]$ with $K[Z]/L \simeq K$.

So far, we have seen one example where, if K is an algebraically closed field, then $K^t[G]$ is diagonally equivalent to $K[G]$, namely, when G is a cyclic group. We state two more results that give conditions in order for this to happen.

Lemma 1.15. Let $K^t[G]$ be given with K algebraically closed. Set $U = \{a\bar{x} | a \in K - \{0\}, x \in G\}$, and let $W = \{a\bar{1} | a \in K - \{0\}\}$. Then $K^t[G]$ is diagonally equivalent to $K[G]$ if and only if U', the commutator subgroup of U, is disjoint from W.

As an immediate consequence, we obtain the following result.

Corollary 1.16. Let K be an algebraically closed field, and let G be abelian. Then $K^t[G]$ is diagonally equivalent to K[G] if and only if $K^t[G]$ is commutative.

A final example, in which $K^t[G]$ is always K[G], is provided by the case where K is a perfect field of characteristic p and G is a finite p-group.

We remark that the material discussed so far in this section has been taken from Passman [22]. Next we give some new results that we have obtained in order to be able to multiply elements of a twisted group ring over a finite cyclic group.

In the remainder of this section we let $G = \langle g \rangle$ be a cyclic group of prime order q, unless otherwise specified.

Definition. $\pi_a = \prod_{h=1}^{q-1} \gamma(g^a,g^h)$, where $1 \le a \le q$.

Notice that since γ is normalized, $\pi_q = 1$ and $\pi_a = \pi_a\gamma(g^a,g^q)$.

Proposition 1.17. $\pi_i\pi_j = \pi_{i+j}\gamma(g^i,g^j)^q$, where $1 \le i,j \le q-1$.

Proof. Putting $x = g^i$, $y = g^j$ and $z = g^k$ in equation (2), we have

$$(5) \quad \gamma(g^i,g^{j+k})\gamma(g^j,g^k) = \gamma(g^{i+j},g^k)\ \gamma(g^i,g^j).$$

Fixing i and j and taking products, we get

$$\prod_{k=2}^{q-1} \gamma(g^i,g^{j+k})\gamma(g^j,g^k) = \prod_{k=2}^{q-1} \gamma(g^{i+j},g^k)\gamma(g^i,g^j);$$

hence

$$[\prod_{k=2}^{q-1}\gamma(g^i,g^{j+k})][\prod_{k=2}^{q-1}\gamma(g^j,g^k)] = [\prod_{k=2}^{q-1}\gamma(g^{i+j},g^k)]\gamma(g^i,g^j)^{q-2}.$$

It follows that

$$\frac{\pi_i}{\gamma(g^i,g^j)\gamma(g^i,g^{j+1})} \cdot \frac{\pi_j}{\gamma(g^j,g)} = \frac{\pi_{i+j}}{\gamma(g^{i+j},g)}\gamma(g^i,g^j)^{q-2}.$$

Then

$$\pi_i\pi_j = \pi_{i+j}\frac{\gamma(g^i,g^j)^{q-1}\gamma(g^i,g^{j+1})\gamma(g^j,g)}{\gamma(g^{i+j},g)} .$$

By letting $k = 1$ in (5), we get $\gamma(g^i,g^{j+1})\gamma(g^j,g) = \gamma(g^{i+j},g)\gamma(g^i,g^j)$. Therefore, $\pi_i\pi_j = \pi_{i+j}\,\gamma(g^i,g^j)^q$, where $1 \le i,j \le q-1$.

Lemma 1.18. Let $\gamma(g^i,g^k)$ be given for $k = 1,2,\ldots,q-1$. If $1 \le x \le q-1$, $x \ne i$, then

$$\pi_x = \gamma(g^i,g^{q-i})^q\gamma(g^i,g^{q-2i})^2 \cdots \gamma(g^i,g^{q-ni})^q/\pi_i^n ,$$

where n is the smallest positive integer such that $x \equiv -ni \pmod{q}$.

Proof. First, we claim that if $1 \le x \le q-1$, then $x \equiv q-ni \pmod{q}$, where $1 \le n \le q-1$. To prove this is enough to show that any two elements of the set $\{q-ni\ 0 \le n \le q-1\}$ represent different residual classes of Z_q. For contradiction, suppose that $q-n_1i \equiv q-n_2i \pmod{q}$, where $0 \le n_1 < n_2 \le q-1$. Then $n_1 \equiv n_2 \pmod{q}$, which contradicts $0 < n_2-n_1 < q$.

Let $x \equiv q-ni \pmod q$. By setting $j = q-i$ in Proposition 1.17, we get $\pi_{q-i} = \gamma(g^i,g^{q-i})^q/\pi_i$ since $\pi_q = 1$. Hence, if we let $j = q-2i$ Proposition 1.17 yields $\pi_{q-2i} = \pi_{q-i}\gamma(g^i,g^{q-2i})^q/\pi_i = \gamma(g^i,g^{q-i})^q\gamma(g^i,g^{q-2i})^q/\pi_i^2$. Thus the result follows by recursion.

Corollary 1.19. If $K = GF(2^n)$, where n is a positive integer such that $2^n - 1 = q$, then $\pi_i\pi_j = \pi_{i+j}$. Moreover, if $\pi_i = 1$ for some i, $1 \leq i \leq q-1$, then $\pi_j = 1$ for all j.

Proof. By hypothesis K is a finite field of order $q+1$; hence its group of units is cyclic of order q. It follows that $\gamma(g^i,g^j)^q = 1$ for all i,j. Then by Proposition 1.17 $\pi_i\pi_j = \pi_{i+j}$. If, moreover, $\pi_i = 1$, then the conclusion follows from Lemma 1.18.

Next we determine some conditions that force the twist function to be trivial.

Proposition 1.20. If $\gamma(g^i,g^h) = 1$ for $h = 1,2,\ldots,q-1$ and i fixed, then γ is trivial.

Proof. By the hypothesis and equation (5) we get $\gamma(g^j,g^k) = \gamma(g^{i+j},g^k)$ with $1 \leq j,k \leq q-1$. As in Lemma 1.18, $j \equiv q-ni \pmod q$ for some n. We use induction on n. If $n = 1$, then $\gamma(g^{q-i},g^j) = \gamma(i,g^k) = 1$ for all k. Assume that $\gamma(g^{q-ti},g^k) = 1$ for all k. Then $\gamma(g^{q-(t+1)i},g^k) = \gamma(g^{q-ti},g^k) = 1$ for all k. Therefore $\gamma(g^j,g^k) = 1$ for all j and k.

Lemma 1.21. If G is a cyclic group generated by g, then $\gamma(g^n,g^m) = \gamma(g^m,g^n)$ for all n,m.

Proof. Since $\bar{g}^n = \prod_{i=1}^{n-1} \gamma(g,g^i)g^{\overline{n}}$ and $\bar{g}^m = \prod_{j=1}^{m-1} \gamma(g,g^j)g^{\overline{m}}$,

then

$$\bar{g}^n \cdot \bar{g}^m = \left(\prod_{i=1}^{n-1} \gamma(g,g^i)g^{\overline{n}}\right)\left(\prod_{j=1}^{m-1} \gamma(g,g^j)g^{\overline{m}}\right) =$$

$$= \left(\prod_{i=1}^{n-1} \gamma(g,g^i)\right)\left(\prod_{j=1}^{m-1} \gamma(g,g^j)\right)\gamma(g^n,g^m)g^{\overline{n+m}}$$

and

$$\bar{g}^m \cdot \bar{g}^n = \left(\prod_{j=1}^{m-1} \gamma(g,g^j)g^{\overline{m}}\right)\left(\prod_{i=1}^{n-1} \gamma(g,g^i)g^{\overline{n}}\right) =$$

$$= \left(\prod_{j=1}^{m-1} \gamma(g,g^j)\right)\left(\prod_{i=1}^{n-1} \gamma(g,g^i)\right)\gamma(g^m,g^n)g^{\overline{m+n}} .$$

Then $\bar{g}^n\bar{g}^m = \bar{g}^{n+m} = \bar{g}^m\bar{g}^n$ implies that $\gamma(g^n,g^m) = \gamma(g^m,g^n)$ for all m,n.

Lemma 1.22. If $1 \le x \le q-1$, then

$$\gamma(g^x,g^x) = \gamma(g,g^x)\gamma(g,g^{x+1})\ldots\gamma(g,g^{2x-1})/\gamma(g,g)\gamma(g,g^2)\ldots \gamma(g,g^{x-1}).$$

Proof. Fix $i = 1$ and $k = x$ in equation (5), and first let $j = x$. Then $\gamma(g,g^{2x})\gamma(g^x,g^x) = \gamma(g^{x+1},g^x)$; hence $\gamma(g^x,g^x) = \gamma(g,g^x)\gamma(g^{x+1},g^x)/\gamma(g,g^2)$. If $j = x + 1$, then equation (5) yields $\gamma(g^{x+1},g^x) = \gamma(g,g^{x+1})\gamma(g^{x+2},g^x)/\gamma(g,g^{2x+1})$. Thus $\gamma(g^x,g^x) = \gamma(g,g^x)\gamma(g,g^{x+1})\gamma(g^{x+2},g^x)/\gamma(g,g^{2x})\gamma(g,g^{2x+1})$. For induction, assume that

$$\gamma(g^x,g^x) = \frac{\gamma(g,g^x)\gamma(g,g^{x+1})\cdots\gamma(g,g^{x+m-1})}{\gamma(g,g^{2x})\gamma(g,g^{2x+1})\cdots\gamma(g,g^{2x+m-1})}\ \gamma(g^{x+m},g^x).$$

Letting $j = x+m$ in equation (5), we get

$$\gamma(g^{x+m},g^x) = \frac{\gamma(g,g^{x+m})}{\gamma(g,g^{2x+m})}\ \gamma(g^{x+m+1},g^x).$$

Hence

$$\gamma(g^x,g^x) = \frac{\gamma(g,g^x)\gamma(g,g^{x+1})\cdots\gamma(g,g^{x+m})}{\gamma(g,g^{2x})\gamma(g,g^{2x+1})\cdots\gamma(g,g^{2x+m})}\ \gamma(g^{x+m+1},g^x).$$

Finally, since for $j = q-1$, we have

$$\gamma(g^{q-1},g^x) = \frac{\gamma(g,g^{q-1})}{\gamma(g,g^{x+q-1})}\ \gamma(g^q,g^x) = \frac{\gamma(g,g^{q-1})}{\gamma(g,g^{x-1})}\ .$$ Thus

$$\begin{aligned}\gamma(g^x,g^x) &= \frac{\gamma(g,g^x)\gamma(g,g^{x+1})\cdots\gamma(g,g^{q-1})}{\gamma(g,g^{2x})\gamma(g,g^{2x+1})\cdots\gamma(g,g^{x-1})}\\ &= \frac{\pi_1/\gamma(g,g)\gamma(g,g^2)\cdots\gamma(g,g^{x-1})}{\pi_1/\gamma(g,g^x)\gamma(g,g^{x+1})\cdots\gamma(g,g^{2x-1})}\\ &= \frac{\gamma(g,g^x)\gamma(g,g^{x+1})\cdots\gamma(g,g^{2x-1})}{\gamma(g,g)\ \ (g,g^2)\cdots\gamma(g,g^{x-1})}\quad .\end{aligned}$$

Proposition 1.23. Let $K = GF(2^n)$, where $2^{n-1} = q$ is a prime. If $1 \le p \le q-1$, then

$$\gamma(g^p,g^p)^{2^{n-1}}\gamma(g^{2p},g^{2p})^{2^{n-2}}\cdots\gamma(g^{2^k p},g^{2^k p})^{2^{n-k-1}}\cdots$$
$$\gamma(g^{2^{n-1}p},g^{2^{n-1}p}) = \pi_1^p\ .$$

Proof. For $b,c \in Z$, $b < c$, let $[b,c]$ denote the product $\gamma(g,g^b)\gamma(g,g^{b+1})\cdots\gamma(g,g^c)$. By Lemma 1.22 we see that $\gamma(g^{2^k p},g^{2^k p}) = [2^k p,2^{k+1}p-1]/[1,2^k p-1]$ for $k = 0,1,\ldots,n-1$. Hence

$$(6)\quad \gamma(g^p,g^p)^{2^{n-1}}\gamma(g^{2p},g^{2p})^{2^{n-2}}\cdots\gamma(g^{2^{n-1}p},g^{2^{n-1}p}) =$$

$$\left(\frac{[p,2p-1]}{[1,p-1]}\right)^{2^{n-1}}\left(\frac{[2p,2^2p-1]}{[1,2p-1]}\right)^{2^{n-2}}\cdots\left(\frac{[2^{n-1}p,2^np-1]}{[1,2^{n-1}p-1]}\right) =$$

$$\frac{[p,2p-1]^{2^{n-1}}[2p,2^2p-1]^{2^{n-2}}\cdots[2^{n-2}p,2^{n-1}p-1]^2[2^{n-1}p,2^np-1]}{[1,p-1]^{\sum_{i=0}^{n-1}2^i}[p,2p-1]^{\sum_{i=0}^{n-2}2^i}[2p,2^2p-1]^{\sum_{i=0}^{n-3}2^i}\cdots[2^{n-2}p,2^{n-1}p-1]}.$$

Since $\sum_{i=0}^{n-a}2^i = 2^{n-a+1}-1$ for $a < n$, we get the expressions in equation (6) equal to

$$(7)\quad \frac{[p,2p-1]^{2^{n-1}}[2p,2^2p-1]^{2^{n-2}}\cdots[2^{n-2}p,2^{n-1}p-1]^2[2^{n-1}p,2^np-1]}{[1,p-1]^{2^n-1}[p,2p-1]^{2^{n-1}-1}[2p,2^2p-1]^{2^{n-2}-1}\cdots[2^{n-2}p,2^{n-1}p-1]}.$$

But from the hypothesis the multiplicative subgroup of K is cyclic of order 2^n-1; hence $[1,p-1]^{2^n-1} = 1$. Thus by cancellation (7) becomes $[p,2p-1][2p,2^2p-1]\cdots[2^{n-1}p,2^np-1] = [p,2^np-1]$. Now for all $b \in Z$, $[b,b+q-1] = \gamma(g,g^b)\gamma(g,g^{b+1})\cdots\gamma(g,g^{b+q-1}) = \gamma(g,g^b)\gamma(g,g^{b+1})\cdots\gamma(g,i)\gamma(g,g)\cdots\gamma(g,g^{b-1}) = \pi_1$. Since $[p,2^np-1]$ is a product of $(2^np-1) - (p-1) = (2^n-1)p = qp$ factors, then $[p,2^np-1] = [p,p+q-1][p+q,p+2q-1]\cdots[p+(p-1)q,p+pq-1] = \pi_1^p$. Therefore the result follows.

Corollary 1.24. Under the hypothesis of Proposition 1.23, if $\pi_j = 1$ for some j, $1 \le j \le q-1$, then $\gamma(g^p,g^p)^{2^{n-1}}\gamma(g^{2p},g^{2p})^{2^{n-2}}\cdots\gamma(g^{2^{n-1}p},g^{2^{n-1}p}) = 1$.

Proof. This follows directly from Corollary 1.19 and Proposition 1.23.

1.5 Twisted Semigroup Rings

Let R be a ring with identity and let S be a multiplicative semigroup. Let γ be a twist function from $S \times S$ into the group U of central units of R; that is, $\gamma(x,y)\gamma(xy,z) = \gamma(y,z)\gamma(x,yz)$ for all $x,y,z \in S$. We define a twisted semigroup ring, denoted by $R^t[S]$, as the R-algebra with basis $\{\bar{s} | s \in S\}$ and with multiplication defined distributively such that, for all s_1,s_2 in S,

$$\bar{s}_1\bar{s}_2 = \gamma(s_1,s_2)\overline{s_1s_2} .$$

Since

$$(r_1\bar{s}_1\cdot r_2\bar{s}_2)r_3\bar{s}_3 = r_1r_2\gamma(s_1,s_2)\overline{s_1s_2}\cdot r_3\bar{s}_3 =$$

$$r_1r_2\gamma(s_1,s_2)r_3\gamma(s_1s_2,s_3)\overline{s_1s_2s_3}$$

and

$$r_1\bar{s}_1(r_2\bar{s}_2\cdot r_3\bar{s}_3) = r_1\bar{s}_1\cdot r_2r_3\gamma(s_2,s_3)\overline{s_2s_3} =$$

$$r_1r_2r_3\gamma(s_2,s_3)\gamma(s_1,s_2s_3)\overline{s_1s_2s_3}$$

then $r_1r_2\gamma(s_1,s_2)r_3\gamma(s_1s_2,s_3) = r_1r_2r_3\gamma(s_2,s_3)\gamma(s_1,s_2s_3)$

for all $r_1, r_2, r_3 \in R$ and $s_1, s_2, s_3 \in S$. This shows that the condition that γ be defined into the central units of R is needed in order to preserve the associativity.

From now on we will abuse in our notation by identifying the elements of S (of a group G) with their images in $R^t[S]$ ($R^t[G]$) under the natural embedding; that is, we will write $s(g)$ for $\bar{s}(\bar{g})$.

Let A be an ideal of R. The set $B = \{\sum_{i=1}^{n} a_i s_i \mid a_i \in A, s_i \in S, n \in Z^+\}$ is an ideal of $R^t[S]$ that we will denote by the (abusive) descriptive symbol $A^t[S]$. If the semigroup S has an identity, then the ideal $A^t[S]$ is the extension of the ideal A of R to $R^t[S]$; that is, $AR^t[S] = A^t[S]$. Moreover, $AR^t[S] \cap R = A$.

Let X be an ideal of S. Then $R^t[X]$ is an ideal of $R^t[S]$, where the twist function of $R^t[X]$ is the restriction to $X \times X$ of the twist function of $R^t[S]$.

The following results are generalizations of similar results in [20] for semigroup rings.

Lemma 1.25. If $\{A_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals of the ring R, then $(\cap_{\lambda\in\Lambda} A_\lambda)^t[S] = \cap_{\lambda\in\Lambda}(A_\lambda^t[S])$.

Theorem 1.26. Let R_1 and R_2 be rings with identity, and let $\theta : R_1 \to R_2$ be a homomorphism of R_1 onto R_2. Let S be a semigroup, and let $\gamma_i : S \times S \to R_i$, i=1,2, be twist functions such that $\theta\gamma_1 = \gamma_2$. Then there is a unique homomorphism $\bar{\theta} : R_1^t[S] \to R_2^t[S]$ from the ring $R_1^t[S]$ with twist function γ_1 onto the ring $R_1^t[S]$ with twist function

γ_2 such that $\bar{\theta}(rs) = \theta(r)s$ for each $r \in R_1$ and $s \in S$. Moreover if A is the kernel of θ, then the kernel of θ is $A^t[S]$.

It follows that if A is an ideal of the ring R, then $R^t[S]/A^t[S] \cong (R/A)^t[S]$. Moreover, θ is one-to-one if and only if $\bar{\theta}$ is one-to-one.

Finally, in the remainder of this dissertation, we denote the nilpotent, prime and Jacobson radicals of R by N, P and J respectively; we maintain our previous notation for these radicals whenever we deal with twisted or ordinary semigroup rings.

CHAPTER II

ON NONTRIVIAL UNITS OF TWISTED GROUP RINGS AND TWISTED SEMIGROUP RINGS

Higman [12] defines a trivial unit of a group ring R[G] as any element of the form rg, where r is a unit of R and $g \in G$. He also characterizes the group rings that have always nontrivial units in the case where R is the ring of the rational integers and G is a torsion group. Since then, the problem of the existence of nontrivial units in group rings, together with the characterization of the group that they determine, has been largely studied. We only mention a few such results. Ayoub and Ayoub [1] determined the group of units of ZG, where G is a finite abelian group. Raggi Cardenas [23] calculates the group of units of K[G] for any Galois field K and any finite abelian group G. Passman [21, Lemma 26.1] shows that if K is any field and G is a nontorsion free group, then K[G] has nontrivial units provided $|K| > 3$. Finally, Gilmer and Teply [11] have shown that if S is a torsion semigroup and R[S] has no nontrivial units, then the characteristic of R is 2,3 or 0; moreover, if R is an algebra over a field, if S is torsion and if R[S] has no nontrivial units, then the characteristic of R is either 2 or 3. Finally, they characterize the semigroup rings of characteristic 2 or 3 which have no nontrivial units.

Let K be a field, and let G be a nontorsion free group. Moreover, let R be a ring with identity, and let S be a semigroup with identity i. In this chapter we answer the following questions:

(A) When does the twisted group ring have nontrivial units?

(B) What are some necessary conditions for $R^t[S]$ to have no nontrivial units?

(C) If S is a nontorsion free semigroup and R has finite characteristic, when does $R^t[S]$ have nontrivial units?

(D) If S is a nontorsion free semigroup, when does $K^t[S]$ have nontrivial units?

In section 2.1 we answer (A) by showing that $K^t[G]$ has always nontrivial units, provided that the twist function is nonconstant.

Question (B) is answered in Theorem 2.9, which states that if $R^t[S]$ has no nontrivial units, then R has no nilpotent elements and either R is indecomposable or the largest subgroup of S containing i is trivial.

In Theorem 2.15 we prove that if S is a nontorsion free semigroup and R has finite characteristic, then $R^t[S]$ has nontrivial units, provided that the twist is nonconstant. This answers question (C).

Theorem 2.16 answers question (D) by showing that if S is nontorsion free and the twist function is nonconstant, then $K^t[S]$ has nontrivial units.

Passman [22] proves the existence of nontrivial units in K[G] by showing first that these group rings always have proper zero divisors. This is why, though independent of of the chapter's title, in section 2.4 we study proper zero divisors in some twisted group rings and give some sufficient conditions for their existence.

2.1 Nontrivial Units in Twisted Group Rings

Let K be a field, and let G be a nontorsion free group. We denote by i the identity of G, and by char K the characteristic of K. Passman [22, Lemma 13.1.1] has proven that with the exceptions of $K = GF(2)$, $|G| = 2$ or 3, and $K = GF(3)$, $|G| = 2$, $K[G]$ has nontrivial units. Our goal in this section is to extend this result to twisted group rings by showing that $K^t[G]$ has nontrivial units whenever the twist function is nonconstant.

Lemma 2.1. Let K be a field of characteristic distinct from 2. Let G be a nontorsion free group. Then the twisted group ring $K^t[G]$ always has nontrivial units, provided that the twist function is nonconstant.

Proof. Since G is nontorsion free and $|G| > 1$, it contains a subgroup H of prime order. Moreover, since any unit of $K^t[H]$ is clearly a unit of $K^t[G]$, then it is sufficient to prove the result for $G = C_q$, where C_q is the cyclic group of order q generated by g.

Let the characteristic of K be p, where either $p = 0$ or else p is a prime integer different from 2.

$$\text{Let } \pi_1 = \prod_{i=1}^{q-1} \gamma(g,g^i).$$

$$\text{Set } b_{k-1} = \prod_{i=1}^{k-1} \gamma(g,g^i) \text{ for } k \geq 2.$$

Notice that $b_{q-1} = b_q = \pi_1$, and $g(b_{k-1}g^k) = b_k g^{k+1}$ for $k \geq 2$. We consider two cases.

First, if $\pi_1 \not\equiv 1 \pmod{p}$, then $1 - \pi_1 \not\equiv 0 \pmod{p}$; hence $1 - \pi_1$ has an inverse. Thus for $q > 2$, we have

$$[(1-\pi_1)^{-1}(i-g)](i + g + \sum_{k=2}^{q-1} b_{k-1}g^k) =$$

$$(1-\pi_1)^{-1}(i + g + \sum_{k=2}^{q-1} b_{k-1}g^k - g - \sum_{k=2}^{q-1} b_{k-1}g^k - \pi_1 i) =$$

$$(1-\pi_1)^{-1}(1-\pi_1)i = i.$$

If $q = 2$, $[(1-\pi_1)^{-1}(i-g)](i+g) = (1-\pi_1)^{-1}(i-g+g-\pi_1 g^2) = (1-\pi_1)^{-1}(1-\pi_1)i = i$.

Secondly, if $\pi_1 \equiv 1 \pmod{p}$, then $p \neq 2$ implies that there exists $(1 + \pi_1)^{-1}$. Thus

$$[(1 + \pi_1)^{-1}(i+g)](i+\pi_1 g + \sum_{\substack{k=2n\\n=1}}^{\frac{q-1}{2}} b_{k-1}g^k - \sum_{\substack{k=2n+1\\n=1}}^{\frac{q-3}{2}} b_{k-1}g^k) =$$

$$(1 + \pi_1)^{-1}(i-\pi_1 g + \sum_{\substack{k=2n\\n=1}}^{\frac{q-1}{2}} b_{k-1}g^k - \sum_{\substack{k=2n+1\\n=1}}^{\frac{q-3}{2}} b_{k-1}g^k + g - \pi_1 \sum_{\substack{k=2n\\n=1}}^{\frac{q-1}{2}} b_{k-1}g^k +$$

$$\sum_{\substack{k=2n+1\\n=1}}^{\frac{q-3}{2}} b_{k-1}g^k + \pi_1 i) = (1 + \pi_1)^{-1}[(1 + \pi_1)i + (1 - \pi_1)(g +$$

$$\sum_{\substack{k=2n\\n=1}}^{\frac{q-1}{2}} b_{k-1}g^k)] = i.$$

Remark 2.2. Notice that by using π_i $(i \leq q - 1)$ instead of π_1 in the proof of Lemma 2.1, we get the same kind of units.

The following result provides an example where the twisted group ring has nontrivial units and the usual group ring does not.

Corollary 2.3. If $K = GF(3)$ and $G = C_2$, then $K^t[G]$ has nontrivial units whenever the twist function is nonconstant.

Next, we consider the case of fields of characteristic 2.

Lemma 2.4. Let K be a field of characteristic 2, and let G be a nontorsion free group. If the twist function is nonconstant, then $K^t[G]$ has always nontrivial units.

Proof. As in Lemma 2.1, it is sufficient to consider G to be a cyclic group of prime order; say $G = C_q = \langle g \rangle$.

Assume first that $q > 2$. Let π_1 and b_k be as defined in the proof of Lemma 2.1. We consider two cases.

If $\pi_1 \not\equiv 1 \pmod 2$, then by proceeding as in Lemma 2.1, we get

$$[(1 - \pi_1)^{-1}(i-g)](i+g+\sum_{k=2}^{q-1} b_{k-1}g^k) = i.$$

Now let $\pi_1 \equiv 1 \pmod 2$.

Suppose, first, that there exists $r \in K - \{0\}$ such that $r^q \neq 1$; that is, $1 + r^q$ has inverse. Then

$$(1+r^q)^{-1}(i+rg)\ (i+rg+\sum_{k=2}^{q-1} r^k b_{k-1} g^k) =$$

$$(1+r^q)^{-1}(i+rg+\sum_{k=2}^{q-1} r^k b_{k-1} g^k+rg+\sum_{k=2}^{q-1} r^k b_{k-1} g^k+r^q\pi_1 i) =$$

$$(1+r^q)^{-1}(1+r^q)i = i.$$

Suppose next that $r^q = 1$ for all $r \in K - \{0\}$. Then K is a finite field. Since char $K = 2$, then $K = GF(2^n)$, where $2^n - 1 = q$. By Corollary 1.19 $\pi_i = 1$ for all i, $1 \le i \le q-1$, so we cannot use the construction of the case where $\pi_1 \ne 1$. To prove that in this case $K^t[G]$ has nontrivial units, it is sufficient to show that, for all $v \in K - \{0\}$, there exists $\alpha \in K^t[G]$ such that $\alpha^2 = v\alpha$. For then we can choose $x,y \in K - \{0\}$ with $x + y \ne 0$, and set $v = (x+y)/xy$. Then $xy\alpha^2 = (x+y)\alpha$; hence

$$(i - x\alpha)(i - y\alpha) = i - (x+y)\alpha + xy\alpha^2 = i.$$

Let $\alpha = \sum_{i=1}^{q-1} a_i g^i \in K^t[G]$. By Lemma 1.21, $\gamma(g^u,g^w) = \gamma(g^w,g^u)$ for all $u,w \in Z$. Thus since by hypothesis char $K = 2$, then

$$\alpha^2 = (\sum_{i=1}^{q-1} a_i g^i)^2 = \sum_{i=1}^{m} a_i^2\gamma(g^i,g^i)g^{2i} + \sum_{j=1}^{m} a_{m+j}^2\gamma(g^{m+j},g^{m+j})g^{2j-1},$$

where $m = q-1/2$. For $\alpha^2 = v\alpha$ we must have

$$(1)\quad a_i^2\ \gamma(g^i,g^i) = \begin{cases} va_{2i}, & \text{if } 1 \le i \le \frac{q-1}{2} \\ va_{2j-1}, & \text{if } \frac{q-1}{2} + 1 \le i \le q-1, \text{ where} \\ j = i - \frac{q-1}{2}\ . \end{cases}$$

Now let $Q = \{1,2,\ldots,q-1\}$ and define $x \sim y$ if $y \equiv 2^h x \pmod{2^n-1}$, where $x,y \in Q$, $h \in Z$. Clearly $\sim$ is an equivalence relation. Since $2^n p \equiv p \pmod{2^n-1}$ for all $p \in Q$, then each of the equivalence classes that $\sim$ determine on Q has the form $\{2^u p \mid 1 \le u \le n\}$. Thus we have Q as the union of $(q-1)/n$ mutally disjoint subsets of n elements.

Notice first, that the system (1) has q-1 unknowns and as many equations. Moreover, any unknown a_p is related to a_{2p}; in general, $a_{2^k p}$ is related to $a_{2^{k-1}p}$ in one equation and to $a_{2^{k+1}p}$ in another. Hence, by the previous argument the set $\{a_x \mid x \in Q\}$ of unknowns is partioned into $(q-1)/n$ subsets of the form $\{a_{2^u p} \mid 1 \le u \le n\}$ for various p. Since the unknowns of each of the q-1 equations are contained in one and only one of these subsets, then we have obtained $(q-1)/n$ systems of n equations each of which has the form:

$$(2) \quad v = \frac{a_p^2 \gamma(g^p, g^p)}{a_{2p}} = \cdots = \frac{a_{2^k p}^2 \gamma(g^{2^k p}, g^{2^k p})}{a_{2^{k+1}p}} = \cdots = \frac{a_{2^{n-1}p}^2 \gamma(g^{2^{n-1}p}, g^{2^{n-1}p})}{a_p},$$

where $1 \le k \le n-1$. Now let, for instance, $a_p = cv$, where $c \in K - \{0\}$. Then we consider the set:

$$a_p = cv$$

$$a_{2p} = c^2 v\gamma(g^p,g^p)$$

$$\vdots$$

$$a_{2^k p} = c^{2^k} v\gamma(g^p,g^p)^{2^{k-1}} \gamma(g^{2p},g^{2p})^{2^{k-2}} \ldots \gamma(g^{2^{k-2}p},g^{2^{k-2}p})$$

$$\vdots$$

$$a_{2^{n-1}p} = c^{2^{n-1}} v\gamma(g^p,g^p)^{2^{n-2}} \gamma(g^{2p},g^{2p})^{2^{n-3}} \ldots \gamma(g^{2^{n-2}p},g^{2^{n-2}p}).$$

We claim that, in fact, this is a set of solutions for (2). By direct substitution we see that they satisfy the first n-1 equations of (2). Thus all we need to show is that

$$v = \frac{a^2_{2^{n-1}p}\,\gamma(g^{2^{n-1}p},g^{2^{n-1}p})}{a_p}\,;$$

that is,

$$v = c^{2^n} v^2 \gamma(g^p,g^p)^{2^{n-1}} \gamma(g^{2p},g^{2p})^{2^{n-2}} \ldots \gamma(g^{2^{n-1}p},g^{2^{n-1}p})/cv.$$

But $c^{2^{n-1}} = 1$, and thus the equality follows from Corollary 1.24.

This shows that each of these systems (2) of equations and hence (1) has a solution. In fact we obtain one set of solutions for each of the q possible values of c. Thus the existence of α is proven.

Finally, let $q = 2$; that is, let $G = \{i,g\}$. Since γ is normalized and nonconstant, $\gamma(g,g) \neq 1$. Hence there exists $(1-\gamma(g,g))^{-1}$, and then $[(1-\gamma(g,g))^{-1}(i+g)]$ is a nontrivial unit of $K^t[G]$ with inverse $i-g$.

We can now put the pieces together to get the following result.

Theorem 2.5. If G is a nontorsion free group and K is a field, then $K^t[G]$ always has nontrivial units, provided that the twist function is nonconstant.

Proof. This follows directly from Lemmas 2.1 and 2.4.

2.2 Nontrivial Units In Twisted Semigroup Rings

Let S be a semigroup (possibly infinite) with identity i. Denote by G the largest subgroup of S containing i. Let R be a ring with identity. In this section we find conditions necessary to insure that the twisted semigroup ring $R^t[S]$ does not have nontrivial units.

Notice that in this section we have the twist function normalized but not necessarily nonconstant.

Lemma 2.6. If R contains a nonzero nilpotent element, then $R^t[S]$ has nontrivial units.

Proof. Let a be a nonzero nilpotent element of R, and let n be the smallest positive integer such that $a^n = 0$. Set

$$b = \begin{cases} a^{n/2} & \text{if } n \text{ is even,} \\ a^{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

Then b is a nonzero element of R such that $b^2 = 0$. Let $s \in S$ with $s \neq i$. Then $(i + bs)(i - bs) = i$.

Corollary 2.7. Let $R = L_1 \oplus \dots \oplus L_n$ be a finite direct sum of left ideals. If $R^t[S]$ has no nontrivial units, then L_i is a two-sided ideal for $i = 1,2,\dots,n$.

Proof. Let $1 = e_1 + \dots + e_n$, where $e_i \in L_i$. Then the e_i are orthogonal idempotent elements of R such that $L_i = Re_i$. Thus if $r \in R$, $(e_i r e_j)^2 = 0$ for all $i \neq j$. By hypothesis $R^t[S]$ has no nontrivial units; thus Lemma 2.6 implies that $e_i r e_j = 0$ for $i \neq j$. Hence $L_i L_j = R(e_i R e_j) = 0$ for $i \neq j$. Therefore L_i is a two-sided ideal of R for all i.

Lemma 2.8. Let $R^t[S]$ be a twisted semigroup ring. Let $R = \bigoplus_{i=1}^{n} R_i$ be a direct sum of nonzero ideals R_i, $i = 1,\dots,n$. Then there exist $\gamma_i : S \times S \to R_i$, $i = 1,\dots,n$, such that $\gamma(s_1,s_2) = \sum_{i=1}^{n} \gamma_i(s_1,s_2)$ for all $(s_1,s_2) \in S \times S$, and $R^t[S] = \bigoplus_{i=1}^{n} R_i^{t_i}[S]$.

Proof. Recall, first, that γ is defined on the central units of R. Clearly $u = u_1 + u_2 + \dots + u_n$ is a central unit of R if and only if u_i is a central unit of R_i, $i = 1,2,\dots,n$. Let θ_i stand for the canonical epimorphism from R onto R_i. Define $\gamma_i : S \times S \to R_i$ by $\gamma(s_1,s_2) = \theta_i \gamma(s_1,s_2)$. Since γ and θ_i are well defined functions that preserve the associativity, so does γ_i. Moreover, our first observation shows that γ_i is defined onto the central units of R_i. Finally, $\sum_{i=1}^{n} \gamma_i(s_1,s_2) = \sum_{i=1}^{n} \theta_i \gamma(s_1,s_2) = \gamma(s_1,s_2)$ for all $(s_1,s_2) \in S \times S$. It follows that $R^t[S] = \bigoplus_{i=1}^{n} R_i^{t_i}[S]$.

We remark that the functions γ_i defined in the former lemma are normalized with respect to the identities of the respective rings.

Proposition 2.8. Let R be a finite (ring) direct sum of nonzero ideals $R_1,\ldots,R_n$. Then the following conditions are equivalent.

(i) $R^t[S]$ has no nontrivial units.

(ii) Each $R_i^{t_i}[S]$ has no nontrivial units for $i = 1,2,\ldots,n$, and $G = \{i\}$.

Proof. By Lemma 2.7 $R^t[S] = \bigoplus_{i=1}^{n} R_i^{t_i}[S]$; thus every element $\alpha \in R^t[S]$ can be written uniquely as $\alpha = \alpha_1 + \ldots + \alpha_n$ with $\alpha_i \in R_i^{t_i}[S]$. Hence α is a unit of $R^t[S]$ if and only if each α_i is a unit of $R_i^{t_i}[S]$. Write $1 = e_1 + \ldots + e_n$, where e_i is the identity of R_i.

Assume that $R^t[S]$ has no nontrivial units. If $R_j^{t_j}[S]$ has a nontrivial unit v_j for some j, then $e_1 + \ldots + v_j + \ldots + e_n$ is a nontrivial unit of $R^t[S]$. Moreover, if there exists $g \in G$ such that $g \neq i$, then $e_1 g + e_2 + \ldots + e_n$ is a nontrivial unit of $R^t[S]$. Therefore if $R^t[S]$ has no nontrivial units, then each $R_i^{t_i}[S]$ has no nontrivial units, and G is trivial.

The converse follows easily from the fact that if $v = v_1 + \ldots + v_n$ is a nontrivial unit of $R^t[S]$, then for some j, v_j is a nontrivial unit of $R^t[S]$, provided that $G = \{i\}$.

Theorem 2.9. If $R^t[S]$ has no nontrivial units, then R has no nilpotent elements, and either R is indecomposable or G is trivial.

Proof. This follows directly from Lemma 2.6 and Proposition 2.8.

2.3 Nontrivial Units In Twisted Semigroup Rings Over Nontorsion Free Semigroups

Gilmer and Teply [11] have proven that if S is a torsion semigroup and R[S] has no nontrivial units, then the characteristic of R is 2,3 or 0; moreover, if R is an algebra over a field, if S is torsion and if R[S] has no nontrivial units then the characteristic of R is either 2 or 3. Finally, they characterize the semigroup rings of characteristic 2 or 3 that have no nontrivial units.

Let S be a nontorsion free semigroup with identity i, and let G be the largest subgroup of S containing i. Moreover, let R be a ring with identity, and let K be a field. Denote by P the prime subring of R, and denote the characteristic of R by char R. If $s \in S$, then $\langle s \rangle$ stands for the subsemigroup of S generated by s. In this section we extend the previous result to the twisted semigroup ring $R^t[S]$ of finite characteristic and to $K^t[S]$, showing that they have always nontrivial units.

We begin by proving some preliminary results, the first of which is known.

Lemma 2.10. S - G is an ideal of S.

Proof. Let $s_1, s_2 \in S - G$, and suppose that $s_1 s_2 = g \in G$. Then $s_1(s_2 g^{-1}) = i$; hence s_1 is invertible with respect to i and therefore $s_1 \in G$. This contradiction shows that S - G is closed. To show that S - G is an ideal, suppose $sg = g_1 \in G$ for some $s \in S - G$ and $g \in G$. Then $s = g_1 g^{-1} \in G$ which contradicts the choice of s. Hence $(S - G)S \subseteq S - G$, and symmetrically $S(S - G) \subseteq S - G$.

Lemma 2.11. Assume S - G contains a torsion element. Then either char R = 2 or else $R^t[S]$ has a nontrivial unit with support in S - G.

Proof. Let s be a torsion element of S - G. By Theorem 1.1 the subsemigroup $\langle s \rangle$ contains an idempotent e. By Lemma 2.10 $e \in S - G$. Set $b = \gamma(e,e)^{-1}$. Then $(i - 2be)^2 = i + 4b[b\gamma(e,e)-1]e = i$. Hence i - 2be is a nontrivial unit of $R^t[S]$ with support in S - G, unless char R = 2.

Lemma 2.12. Let char R = 2, and let e be an idempotent of S different from i. If the twist function is nonconstant, then e is contained in the support of a nontrivial unit of $R^t[S]$.

Proof. Since γ is nonconstant, then the group of units of R is not trivial. Let $\gamma(e,e) = c$, and let u and v be different units of R. By recalling that c is a central unit of R, then $[ui + c^{-1}(u+v)e][u^{-1}i+c^{-1}(u^{-1}+v^{-1})e] = i+c^{-1}[4+2u^{-1}v+2uv^{-1}]e = i$.

Corollary 2.13. If all the units of $R^t[S]$ are contained in $R^t[G]$, then $S - G$ is torsion free, provided that the twist function is nonconstant.

Proof. Suppose $S - G$ contains a torsion element. If char $R \neq 2$, then by Lemma 2.11 $R^t[S]$ contains a nontrivial unit with support in $S - G$, which contradicts our hypothesis. On the other hand, if char $R = 2$, then by Theorem 1.1 $S - G$ contains an idempotent e. Hence by Lemma 2.12 $R^t[S]$ contains a nontrivial unit with e in its support. Therefore $S - G$ is torsion free.

Proposition 2.14. Let G be a nontorsion free group, and let R have finite characteristic. If the twist function is nonconstant, then $R^t[G]$ has nontrivial units.

Proof. We show that $P^t[G]$ always has nontrivial units. Let char $P = n$, and let $D = \{q_1, q_2, \ldots, q_k\}$ be the set of distinct prime divisors of n. If $k > 1$, then P is decomposable as a finite direct sum of q-rings. Since G is nontorsion free by hypothesis, then Theorem 2.9 implies that $P^t[G]$ has nontrivial units. On the other hand, if $k = 1$, then P is a q-ring. Let $n = q^m$, where $m > 1$. Then P has a nilpotent element; so by Lemma 2.6 $P^t[G]$ has nontrivial units. Finally, if $n = q$, then $P = GF(q)$; hence by Theorem 2.5 $P^t[G]$ contains nontrivial units.

Theorem 2.15. Let S be a nontorsion free semigroup and let R be a ring of finite characteristic with identity. If the twist function is nonconstant, then $R^t[S]$ has nontrivial units.

Proof. Let s be a torsion element of S different from i. If $s \in G$, then G is a nontrivial, nontorsion free group; thus Proposition 2.14 shows that $R^t[G]$ and hence $R^t[S]$ have nontrivial units. Let $s \in S - G$. Then, either char $R \neq 2$ and the result follows from Lemma 2.11, or else char $R = 2$. By Theorem 1.1 and Lemma 2.10, $\langle s \rangle$ contains an idempotent $e \neq i$. Hence, Lemma 2.12 shows that $R^t[S]$ contains a nontrivial unit when char $R = 2$.

If we follow verbatim the proof of Theorem 2.15, except for changing the use of Proposition 2.14 to Theorem 2.5, then we have proven the following result.

Theorem 2.16. Let S be a nontorsion free semigroup and let K be a field. If the twist function is nonconstant, then $K^t[S]$ has nontrivial units.

2.4 Zero Divisors In Twisted Group Rings

It has been proven [22, Lemma 13.1.1] that if G is a nontorsion free group and K is a field, then K[G] has proper zero divisors. In passing to the twisted group rings case, this is no longer true, as we see in the following example. Consider $Q^t[C_4]$, where Q is the field of rational numbers and C_4 is the cyclic group of order 4 generated by g. Let $\pi_1 = \prod_{i=1}^{3} \gamma(g,g^i)$, and assume that $\sqrt{\pi_1} \notin K$. Then $x^4 - \pi_1$ is irreducible; hence

$K^t[C_4]$ is isomorphic to the field $K[x]/(x^4-\pi_1)$. On the other hand, $(i-g^2)(i+g^2) = 0$ gives an easy example of proper zero divisors in $K[C_4]$.

In this section we give some sufficient condtions for the existence of proper zero divisors in $K^t[G]$.

If g is a torsion element of prime order q in G, let

$$\pi_1(g) = \prod_{i=1}^{q-1} \gamma(g,g^i).$$

Theorem 2.17. Let K be a field, and let G be a non-torsion free group. If $\sqrt[q]{\pi_1(g)}\in K$ for some element $g \in G$ of prime order q, then $K^t[G]$ has proper zero divisors.

Proof. Since any proper zero divisor of $K^t[H]$, where H is a subgroup of G, is a zero divisor of $K^t[G]$, it is sufficient to show that $K^t[\langle g\rangle]$ has proper zero divisors for some $g \in G$.

Let $g \in G$ be a torsion element of prime order q, and assume that $\sqrt[q]{\pi_1(g)}\in K$. Set $c = 1/\sqrt[q]{\pi_1(g)}$ and $b_k = \prod_{i=1}^{k-1} \gamma(g,g^i)$. Then

$$(i-cg)(i+cg + \sum_{k=2}^{q-1} c^k b_{k-1} g^k) =$$

$$i+cg + \sum_{k=2}^{q-1} c^k b_{k-1} g^k - cg - \sum_{k=2}^{q-1} c^k b_{k-1} g^k - c^q \pi_1(g)i = (1-c^q\pi_1(g))i=0.$$

Corollary 2.18. If the characteristic of K divides the order of some element of G and the twist function takes values only in the prime field of K, then $K^t[G]$ has proper zero divisors.

Proof. Let char $K = q > 0$. Then the prime field of K is $GF(q)$. Hence $\sqrt[q]{x} \in K$ for all $x \in GF(q)$. By Theorem 2.17 the result follows.

For reasons of completeness we give the following result, the proof of which resembles that of the ordinary group rings.

Lemma 2.19. If G is a torsion free abelian group, then $K^t[G]$ is an integral domain.

Proof. Since G is a torsion free abelian group, then it admits a total order compatible with the product. Let α and β be nonzero elements of $K^t[G]$. Write $\alpha = \sum_{i=1}^{n} a_i x_i$ and $\beta = \sum_{i=1}^{m} b_i y_i$. Since G is ordered, supp α and supp β are totally ordered subsets of G; hence the leading terms of α and β are $a_n x_n$ and $b_m y_m$ respectively. Thus $\alpha\beta$ has a nonzero leading term $a_k b_m \gamma(x_n, y_m) x_n y_m$, which proves the result.

CHAPTER III
NILPOTENT RADICALS OF NONCOMMUTATIVE SEMIGROUP RINGS

Passman [22, theorem 1.9] has determined the nilpotent radical of a group ring over a field of finite characteristic in the following result.

Let K be a field of characteristic $p > 0$ and let G be a group. Then (i) $N[K[G]] = J[K[\Delta^p(G)]] \cdot K[G]$, where $\Delta^p(G) = \langle x \in G |$ the index of the centralizer of x in G is finite and the order of x is a power of $p\rangle$; (ii) $J[K[\Delta^p(G)]] = N[K[\Delta^p(G)]]$; and (iii) $N[K[G]] = 0$ if and only if $\Delta^p(G) = \langle 1 \rangle$.

Gilmer and Parker [10] have completely characterized the nilpotent radical of any commutative semigroup ring in the result that follows.

Let R be a commutative ring and let $\{P_\lambda\}_{\lambda \in \Lambda}$ be the family of prime ideals of R. Let S be an abelian semigroup. Then the nilpotent radical of R[S] is $\cap_{\lambda \in \Lambda} \{P_\lambda[S] + I_{p_\lambda}\}$, where p_λ is the characteristic of R/P_λ and where I_{p_λ} is the ideal of R[S] generated by the set $\{rx^a = rx^b | \, a \underset{p}{\sim} b\}$; if $p_\lambda = 0$, then I_{p_λ} is the ideal of R[S] generated by the set $\{rx^a - rx^b | \, r \in R$ and a is almost equal to $b\}$.

The natural question that arises immediately is whether this result is valid or not in the noncommutative semigroup rings. In the first section of this chapter we analyze an example of a noncommutative semigroup ring over a field, showing that the former result is no longer true.

Then, what is the nilpotent radical of a noncommutative semigroup ring? The rest of this chapter deals with this question, whose answer cannot be unique, since it will depend upon the particular semigroups and rings in consideration. We consider some known families of noncommutative semigroups, and we determine the nilpotent radical of semigroup rings constructed with them.

Thus, in section 3.2 we study the nilpotent radical of a semigroup ring R[S], where R is a commutative ring with identity and S is a right (left) zero semigroup. Theorem 3.6 shows that in this case $N[R[S]] = \{\Sigma a_i x_i \in R[S] \mid \Sigma a_i \in N\}$ where N is the nilpotent radical of R.

In section 3.3 we let S be a right (left) group and R a ring with identity. Theorem 3.19 handles this case by showing that $N[R[S]] = N[R[H_1]]R[S] + C$, where H_1 is a maximal subgroup of S and $C = R[H_1]W$ with $W = \{ \sum_{i=1}^{n} a_i e_i \mid \sum_{i=1}^{n} a_i = 0$, e_i idempotent of S and $n \in Z^+\}$. In section 3.4 we consider the family S of semigroups having a universally minimal ideal, and we let R be a ring with identity. Proposition 3.26 states that $N[R[S]] = N[R[U]] \oplus N[K]$, where U is the group of zeroids of S and K the kernel of the canonical epimorphism $\theta : R[S] \to R[U]$. Then we apply this result to one subfamily of S, where the product outside of U is known.

3.1 An Example

In this section we consider an example of a semigroup ring F[S], where F is a field and S is a torsion free finite semigroup. We calculate the radicals of F[S] to show that by just removing the commutativity of S we may have a nonzero nilpotent radical. This contrasts the following result of Gilmer and Parker [10] on commutative semigroup rings. If either R is a ring of finite characteristic n and S is a p-torsion free semigroup for each prime divisor p of n, or R is a ring of zero characteristic and S is torsion free, then the nilradical of R[S] is N[S], where N is the nilpotent radical of R.

Let S be the semigroup defined by the following table.

$\cdot$	i	x	y
i	i	x	y
x	x	x	y
y	y	x	y

Notice that S has identity element i and is torsion free, but S is not commutative.

Let F be a field. Denote by $\langle x,y\rangle$ the ideal generated by x and y. Let $L = \{ai + bx + cy \mid a,b,c \in F,\ a + b + c = 0\}$. Clearly $L^2 \subseteq L$, $F[S]L \subseteq L$ and $LF[S] \subseteq L$; hence L is an ideal of F[S].

In order to determine the radicals of F[S] we prove the following results.

Claim 1. L and $\langle x,y\rangle$ are maximal ideals of F[S].

Proof. Let $(L,\alpha) = M$ denote the enlargement of L by an element $\alpha = ui + vx + wy$ of $F[S] - L$; that is, $u + v + w \neq 0$. Since $\alpha x = (u + v + w)x \in M$ and since $(u + v + w)(i - x) \in M$, then by adding both elements we get $(u + v + w)i \in M$. But $u + v + w$ has an inverse; hence $i \in M$. Therefore $M = F[S]$.

Similarly, let $(\langle x,y\rangle,\beta) = T$ be any enlargement of $\langle x,y\rangle$ by an element $\beta = ui + vx + wy$ not in $\langle x,y\rangle$, that is, $u \neq 0$. Since $vx + wy \in \langle x,y\rangle$, then ui and hence i belong to T. Therefore $T = F[S]$.

Claim 2. The only ideals of $F[S]$ generated by elements which do not contain i in their support, are $\langle x,y\rangle$ and $\langle x - y\rangle$.

Proof. Let I be an ideal of $F[S]$ such that $i \notin \operatorname{supp}\alpha$ for all $\alpha \in I$. If either x or y belongs to I, then $xy = y$ and $yx = x$ imply that $\langle x,y\rangle = \{ux + vy \mid u,v \in F\} \subseteq I$.

Let $bx + cy \in I$, $b \neq 0$, $c \neq 0$. We consider two cases.

First if $b + c \neq 0$ for some element of I, then $(bx + cy)x = (b + c)x \in I$. But $b + c$ has an inverse; thus $x \in I$, and by the previous paragraph $\langle x,y\rangle \subseteq I$. By Claim 1 $\langle x,y\rangle$ is maximal; hence $I = \langle x,y\rangle$.

On the other hand, if $b + c = 0$ for each element of I, then $bx + cy = b(x - y) \in I$. Hence $\langle x - y\rangle \subseteq I$. Moreover, $\langle x - y\rangle \neq I$ implies that there exist $ax + by \in I$ such that $a + b \neq 0$. This contradiction shows that $x - y = I$.

Claim 3. An element of $F[S]$, which contains i in its support, generates the ideal $F[S]$ if the sum of its coefficients is not zero.

Proof. Let I be an ideal of F[S]. If $ai + bx \in I$, where $a \neq 0$ and $a + b \neq 0$, then $(ai + bx)x = (a + b)x \in I$ implies that $x \in I$. Since $ai = (ai + bx) - bx \in I$, it follows that $i \in I$. Therefore $I = F[S]$. Similarly, if $ai + cy \in I$, where $a \neq 0$ and $a + c \neq 0$, or if $ai + bx + cy \in I$, where $a \neq 0$ and $a + b + c \neq 0$, then $i \in I$. Hence $I = F[S]$.

So far we have found the ideals $\langle x - y\rangle$, $\langle x,y\rangle$ and L. We prove next that, in fact, these are the only proper ideals of F[S].

Claim 4. Any element of F[S], which contains i in its support and which has the sum of its coefficients equal to zero, is a generator of L.

Proof. Let $ai + bx \in L$, where $a \neq 0$ and $a + b = 0$. Then $ai + bx = a(i - x) \in \langle i - x\rangle$. Similarly, if $ai + cy \in L$ with $a \neq 0$ and $a + c = 0$, then $ai + cy = a(i - y) \in \langle i - y\rangle$. Now let $\alpha = ai + bx + cy \in L$, where $a \neq 0$, $b \neq 0$, $c \neq 0$ and $a + b + c = 0$. Since $(i - x)(i - y) = i - x$ and $(i - y)(i - x) = i - y$, then $\langle i - x\rangle = \langle i - y\rangle$. Moreover, from $\alpha(i - x) = \alpha$ and $(i - x)\alpha = a(i - x)$, it follows $\langle i - x\rangle = \langle\alpha\rangle$. Hence, each of the elements of L containing i in its support generates the ideal $\langle i - x\rangle$. By definition, any other nonzero element of L is of the form $bx + cy$ with $b \neq 0$, $b + c = 0$. Then $bx + cy = b(x - y)$. But $x - y = (i - y) - (i - x)$; hence $\langle x - y\rangle \subset \langle i - x\rangle$. Therefore $L = \langle i - x\rangle$.

Since no element of $\langle x - y\rangle$ has i in its support, then $\langle x - y\rangle$ is strictly contained in L. By claims 1 through 4 the lattice of ideals of F[S] is

$$\begin{tikzcd}
 & F[S] \arrow[dl, dash] \arrow[dr, dash] & \\
L \arrow[dr, dash] & & \langle x,y\rangle \arrow[dl, dash] \\
 & \langle x - y\rangle \arrow[d, dash] & \\
 & 0 &
\end{tikzcd} .$$

Since $(x - y)^2 = 0$, then $\langle x - y\rangle$ is a nilpotent ideal. Also $i - x$ and x are idempotent elements of L and $\langle x,y\rangle$, respectively. Hence these ideals are not nilpotent; thus the nilpotent radical of F[S] is $N[F[S]] = \langle x - y\rangle$. Moreover, since S is finite, then F[S] is Artinian. Hence $J[F[S]] = N[F[S]] = \mathcal{P}[F[S]] = \langle x - y\rangle$.

3.2 Semigroup Rings Over Right (Left) Zero Semigroups and Their Nilradicals

In [5] the following definition is given.

A semigroup S is called a right zero semigroup if $xy = y$ for every x,y in S. The left zero semigroups are defined by duality. In spite of their triviality, these semigroups arise naturally in the characterization of some larger families of noncommutative semigroups that we consider in further sections. Let R be a commutative ring with identity, and let S be a right zero semigroup. Our goal in this section is to characterize the nilpotent radical of $R[S]$. But first we need the following four lemmas.

Lemma 3.1. Any right ideal of $R[S]$ is a two-sided ideal.

Proof. Let A be a right ideal of $R[S]$. Hence $AR[S] \subseteq A$. Since R is commutative, then $RA = AR \subset A$. Moreover, $S \cdot \operatorname{supp} \alpha = \operatorname{supp} \alpha$ for all $\alpha \in A$; hence $SA = A$. Then $R[S]A = A$ shows that A is also a left ideal.

Lemma 3.2. Let A be an ideal of R, and let T be a non-empty subset of S. Then $A[T]$ is a left ideal of $R[S]$. Moreover, $A[S]$ is a two-sided ideal of $R[S]$.

Proof. Any subset T of S is a set of right zeros; hence $ST = T$. Thus, $R[S]A[T] \subseteq RA[T] = A[T]$. Also $A[S]S \subseteq A[S]$ and $A[S] \cdot R \subseteq A[S]$ imply that $A[S]R[S] \subseteq A[S]$. Hence $A[S]$ is a right ideal, and the result follows from Lemma 3.1.

Lemma 3.3. Let $\alpha = \Sigma a_i x_i$ be an element of $R[S]$ such that $\Sigma a_i = 0$. Then the principal ideal generated by α, is $(\alpha) = R\alpha$. Moreover, (α) is a nilpotent ideal of index two.

Proof. Since $\alpha x = (\Sigma a_i)x = 0$ for all $x \in S$, then $R\alpha R[S] = 0$; hence $R\alpha$ is a right ideal of $R[S]$. It follows from Lemma 3.1 that $R\alpha$ is a two-sided ideal; that is, $(\alpha) = R\alpha$. Moreover, since $\alpha^2 = (\Sigma a_i x_i)(\Sigma a_i x_i) = (\Sigma a_i)(\Sigma a_i x_i) = 0$ and since R is commutative, then $(R\alpha)(R\alpha) = R\alpha^2 = 0$.

Lemma 3.4. Let R be an integral domain. Let I be a nilpotent ideal of R[S], and let $\alpha = \Sigma a_i x_i \in I$. Then $\Sigma a_i = 0$.

Proof. If $x \in S$, then $\alpha x = (\Sigma a_i)x \in I$. Let n be the index of nilpotency of I. Then $0 = [(\Sigma a_i)x]^n = (\Sigma a_i)^n x$; hence $(\Sigma a_i)^n = 0$. Since R is a integral domain, then $\Sigma a_i = 0$.

Theorem 3.5. Let R be an integral domain. Then the nilpotent radical of R[S] is $N[R[S]] = \{\alpha = \Sigma a_i x_i | \Sigma a_i = 0\}$. Moreover, N[R[S]] is a nilpotent ideal of index two.

Proof. Let $L = \{\Sigma a_i x_i | \Sigma a_i = 0\}$. By Lemma 3.3, if $\alpha \in L$, then $(\alpha) = R\alpha$ is a nilpotent ideal of R[S] of index two. Hence L is a nilpotent ideal of index two.

On the other hand, if I is a nilpotent ideal of R[S], then by Lemma 3.4 $I \subseteq L$; so L is the unique largest nilpotent ideal of R[S]. Thus N[R[S]] = L.

Using this result, we characterize next the nilpotent radical when R is a commutative ring.

Theorem 3.6. Let R be a commutative ring with identity, and let S be a right zero semigroup. Then the nilpotent radical of R[S] is $N[R[S]] = \{\Sigma a_i x_i | \Sigma a_i \in N\}$, where N is the nilpotent radical of R.

Proof. Let $\alpha = \Sigma a_i x_i \in N[R[S]]$. Let $P(R)$ denote the prime radical of R. Since R is commutative, then $N = P(R)$. If P is a prime ideal of R, then R/P is an integral domain.

By Theorem 3.5 $N[(R/P)[S]] = \{\Sigma b_i y_i \mid \Sigma b_i = 0\}$. Let $\theta\colon R[S] \to (R/P)[S]$ be the canonical epimorphism. Since $\theta(\alpha) = \Sigma \bar{a}_i x_i \in N[(R/P)[S]]$, then $\Sigma \bar{a}_i = 0$. Hence $\Sigma a_i + P = P$; that is, $\Sigma a_i \in P$. Since Σa_i belongs to each prime ideal P_λ of R, then $\Sigma a_i \in \cap P_\lambda = N$. Therefore, $N[R[S]] \subseteq \{\Sigma a_i x_i \mid \Sigma a_i \in N\}$.

Conversely, let $\beta = \Sigma b_i y_i$, where $\Sigma b_i \in N$. Since R is commutative, there exists a nilpotent ideal $A \subseteq N$ such that $\Sigma b_i \in A$. If $\rho = \Sigma c_j x_j \in R[S]$, then $\beta\rho = (\sum_i b_i y_i)(\sum_j c_j x_j) = \sum_j \sum_i b_i c_j (x_i y_j) = \sum_j (\sum_i b_i) c_j y_j$. Hence the sum of the coefficients of $\beta\rho$ is $(\sum_i b_i)(\sum_j c_j) \in A$. By a similar argument the sum of the coefficients of any element of $(\beta) = \{\alpha\beta + \beta\rho + n\beta + \Sigma \alpha_i \beta \rho_i \mid \alpha, \rho, \alpha_i, \rho_i \in R[S],\ n \in Z^+\}$ belongs to A. Moreover, if $\delta_i \in (\beta)$ $i = 1, 2, \ldots, n$, then the sum of the coefficients of $\delta_1 \delta_2 \ldots \delta_n$ belongs to A^n. Let k be the index of nilpotency of A. Then the sum of the coefficients of any element of $(\beta)^k$ belongs to $A^k = 0$. It follows now from Lemma 3.3 that $(\beta)^{2k} = 0$. Hence $(\beta) \subseteq N[R[S]]$; and so, $\{\Sigma a_i x_i \mid \Sigma a_i \in N\} \subseteq N[R[S]]$.

Therefore, $N[R[S]] = \{\Sigma a_i x_i \mid \Sigma a_i \in N\}$.

We remark that if S is a left zero semigroup and R is a commutative ring with identity (integral domain), then by duality Theorem 3.6 (Theorem 3.5) also holds in R[S].

Corollary 3.7. Let S be a right (left) zero semigroup, and let R be a commutative ring with indentity. Then R[S] is semiprime if and only if R is a semigroup ring and $|S| = 1$.

Proof. Let R be a semigroup ring, and let $|S| = 1$. Then $R[S] \cong R$; hence $N[R[S]] = 0$.

Conversely, let R be a commutative ring, and let S be a right zero semigroup. Assume that R[S] is semiprime. Then $N[R[S]] = 0$. Hence, if either the nilpotent radical N of R is nonzero, or if $|S| > 1$, then by Theorem 3.6 $N[R[S]] \neq 0$. This contradiction shows that $N = 0$ and $|S| = 1$.

Corollary 3.8. Let R be a commutative ring. Let the semigroup $S = E_1 \times E_2 \times \ldots \times E_n$, where E_i are right zero semigroups. Then the nilpotent radical of R[S] is $N[R[S]] = \bigoplus_{i=1}^{n} \{\sum_j a_{ij}x_{ij} \in R[E_i] \mid \sum_j a_{ij} \in N\}$, where N is the nilpotent radical of R.

Proof. Since $R[S] = \bigoplus_{i=1}^{n} R[E_i]$, then the conclusion follows from Theorem 3.6.

3.3 The Nilpotent Radical of the Semigroup Ring of A Right (Left) Group.

A semigroup S is called a right group if it is right simple and left cancellative [5]. Left groups are defined dually. Being right (left) simple means that the semigroup contains no right (left) proper nonzero ideals; thus these semigroups form a subfamily of the simple semigroups. From now on we consider right groups, but the analogous conclusions for left groups will follow by duality.

Let R be a ring with identity, and let S be a right group. In this section we find the nilpotent ideals of the semigroup ring $R[S]$. This will allow us to completely characterize the nilpotent radical of $R[S]$.

We begin by stating two important results on right groups, the proofs of which can be found in [5, 1.26, 1.27].

Lemma 3.9. Every idempotent element of a right simple semigroup S is a left identity of S.

Lemma 3.10. The following assertions concerning a semigroup S are equivalent.

(i) S is a right group.

(ii) S is right simple and contains an idempotent.

(iii) S is the direct product $G \times E$ of a group G and a right zero semigroup E; that is, there exists $\phi: G \times E \to S: (g,e) \to ge$, such that ϕ is an isomorphism.

Let $E = \{e_i\}_{i \in I}$ stand for the set of idempotents of S. By Lemma 3.10 (ii) E is nonempty. By Lemma 3.9 every element of E is a left identity of S; thus E is a right zero semigroup of S. Let $e \in E$, then $Se = eSe$, and we see that Se is a subgroup of S; in fact, by Lemma 3.10 (iii) $Se \cong G \times \{e\}$. Let H_i denote the subgroup Se_i, where $e_i \in E$. It follows that for any $e_i, e_j \in E$, $i \neq j$, the subgroups H_i and H_j are disjoint and isomorphic under $\theta_{ij}: H_i \to H_j$, where $\theta_{ij}(x_{\lambda i}) = x_{\lambda i} e_j$. Moreover, by Lemma 3.10 (iii) we also see that S is the disjoint union of its subgroups H_i, $i \in I$. We have shown the following restatement of the previous results.

Lemma 3.11. If S is a right group, and $E = \{e_i\}_{i \in I}$ is the set of its idempotent elements, then the following statements hold.

(i) E is a right zero semigroup.

(ii) $H_i = Se_i$ is a subgroup of S for all $e_i \in E$, with identity $1_{H_i} = e_i$.

(iii) If $e_i, e_j \in E$ with $i \neq j$, then $H_i \cap H_j = \phi$ and $\theta_{ij}: H_i \to H_j$ defined by $\theta_{ij}(x_{\lambda i}) = x_{\lambda i}e_j = x_{\lambda j}$ is a group isomorphism.

(iv) $S = \cup_{i \in I} H_i$.

Let R be a commutative ring with identity, and let S be a right group. As a direct consequence of Lemma 3.11 we get the following result.

Lemma 3.12. The following statements hold in R[S].

(i) $(\sum_{i \neq j} R[H_i]) \cap R[H_j] = 0$ for all $j \in I$.

(ii) There exist a group ring isomorphism $\bar{\theta}_{ij}: R[H_i] \to R[H_j]$ such that $\bar{\theta}_{ij}(R[H_i]) = R[H_i]e_j = R[H_j]$ for all $i,j \in I$.

(iii) $R[S] = \overset{w}{\Sigma}_{i \in I} R[H_i]$, a internal weak direct sum of R-modules.

(iv) e_i is a left identity of R[S] for all $i \in I$.

(v) $R[S]e_j = R[H_j]$; that is, each $R[H_j]$ is a left ideal of R[S].

Proof. (i) Follows clearly from $H_i \cap H_j = \phi$ for all $i,j \in I$, $i \neq j$. (ii) Let $\bar{\theta}_{ij}$ be the natural extension of θ_{ij};

that is, define $\bar{\theta}_{ij}: R[H_i] \to R[H_j]$ by $\bar{\theta}_{ij}(\sum_{\lambda=1}^{n} a_{\lambda i}x_{\lambda i}) = \sum_{\lambda=1}^{n} a_{\lambda i}\theta_{ij}(x_{\lambda i}) = \sum_{\lambda=1}^{n} a_{\lambda i}(x_{\lambda i}e_j)$. Since θ_{ij} is a semigroup isomorphism between the basis, then $\bar{\theta}_{ij}$ is a group ring isomorphism. Hence $R[H_j] = \bar{\theta}_{ij}(R[H_i]) = R[H_i]e_j$. (iii) Follows directly from (i) and Lemma 3.11 (iv). (iv) Is clear since e_i is a left identity of S for all $i \in I$. (v) By (iii) and (ii) we get $R[S]e_j = \Sigma_{i\in I}\, R[H_i]e_j = \Sigma_{i\in I}\, R[H_j] = R[H_j]$.

From now on we fix an element of the index set I, say $i \in I$.

Lemma 3.13. Let B_1 be a two-sided ideal of $R[H_1]$. Then $B_i = B_1e_i$ is a left ideal of R[S] for every $i \in I$. Moreover, $B = \Sigma_{i\in I}B_i$ is a two-sided ideal of R[S].

Proof. By Lemma 3.12 (ii) $\bar{\theta}_{ij}$ is a group ring isomorphism for all $i,j \in I$. Hence $\bar{\theta}_{1i}(B_1) = B_1e_i = B_i$ is a two-sided ideal of $R[H_i]$ for all $i \in I$. Since $R[H_j]B_i = R[H_j](e_iB_i) = R[H_i]B_i \subseteq B_i$; then $R[S]B_i = \Sigma_{j\in I}R[H_j]B_i \subseteq B_i$. Thus, B_i is a left ideal of R[S] for all $i \in I$. Therefore, $B = \Sigma_{i\in I}\, B_i$ is a left ideal of R[S]. Moreover, $B_jR[H_i] = B_j(e_iR[H_i]) = B_iR[H_i] \subseteq B_i$ for all $i,j \in I$. Hence $B_jR[S] = B_j\Sigma_{i\in I}\, R[H_i] = \Sigma_{i\in I}\, B_jR[H_i] \subseteq \Sigma_{i\in I}B_i = B$. It follows that $BR[S] = \Sigma_{i\in I}B_iR[S] \subseteq \Sigma_{i\in I}B = B$; thus B is a right ideal of R[S]. Therefore B is a two-sided ideal of R[S].

Proposition 3.14. $\Sigma_{i\in I}\, N[R[H_i]] \subseteq N[R[S]]$.

Proof. Let $N_0[R[H_i]] \subseteq N_1[R[H_i]] \subseteq \cdots \subseteq N_j[R[H_i]] \subseteq \cdots$, $i \in I$, and $N_0[R[S]] \subseteq N_1[R[S]] \subseteq \cdots \subseteq N_j[R[S]] \subseteq \cdots$ be the chains of ideals used in the definition of the nilpotent radical of $R[H_i]$ and $R[S]$ respectively. We use transfinite induction. Let B_1 be a nilpotent ideal of $R[H_1]$. By Lemma 3.12 (ii) $\bar{\theta}_{ij}$ is a ring isomorphism for all $i,j \in I$; hence $\bar{\theta}_{1i}(B_1) = B_1e_i = B_i$ is a nilpotent ideal of $R[H_i]$, $i \in I$, with the same index of nilpotency as B_1, say n. By Lemma 3.13 $B = \Sigma_{i\in I} B_i$ is a two-sided ideal of $R[S]$. We claim that B is nilpotent. Since $B_iB_j = B_i(e_jB_j) = B_j^2$, it follows by induction that $B_{i_1}B_{i_2}\ldots B_{i_m} = B_{i_m}^m$ for all positive integer m. Thus $B_i^n = 0$ for all $i \in I$ implies that $B^n = (\Sigma_{i\in I} B_i)^n = {}_{i\in I} B_i^n = 0$. Therefore $B \subseteq N_0[R[S]]$ and consequently $\Sigma_{i\in I} N_0[R[H_i]] \subseteq N_0[R[S]]$. This shows the first step of the induction argument.

Assume that $\Sigma_{i\in I} N_k[R[H_i]] \subseteq N_k[R[S]]$ for all the ordinals $k < j$. Suppose j is not a limit ordinal and let $A_1/N_{j-1}[R[H_1]]$ be nilpotent in $R[H_1]/N_{j-1}[R[H_1]]$. As in the previous paragraph $A_i/N_{j-1}[R[H_1]] = \bar{\theta}_{1i}[A_1/N_{j-1}[R[H_1]]$ is a nilpotent ideal of $R[H_i]/N_{j-1}[R[H_i]]$, $i \in I$, with the same index of nilpotency, say n. As a clear consequence of Lemma 3.13 $A/\Sigma_{i\in I} N_{j-1}[R[H_i]] = \Sigma_{i\in I} A_i/N_{j-1}[R[H_i]]$ is an ideal of $R[S]/\Sigma_{i\in I} N_{j-1}[R[H_i]]$ which we claim is nilpotent. Since

$$\begin{aligned} A_i/N_{j-1}[R[H_i]] \cdot A_t/N_{j-1}[R[H_t]] &= A_i/N_{j-1}[R[H_i]] \cdot e_tA_t/N_{j-1}[R[H_t]] \\ &= A_t^2/N_{j-1}[R[H_t]], \end{aligned}$$

by induction we obtain

$$A_{i_1}/N_{j-1}[R[H_{i_1}]] \cdot A_{i_2}/N_{j-1}[R[H_{i_2}]] \cdots A_{i_m}/N_{j-1}[R[H_{i_m}]]$$

$$= A^m_{i_m}/N_{j-1}[R[H_{i_m}]].$$

Thus $(A_i/N_{j-1}[R[H_i]])^n = 0$ for all $i \in I$ implies that

$$(A/\Sigma_{i \in I}N_{j-1}[R[H_i]])^n = \Sigma_{i \in I}(A_i/N_{j-1}[R[H_i]])^n = 0.$$

Hence $A/\Sigma N_{j-1}[R[H_i]]$ is nilpotent in $R[S]/\Sigma N_{j-1}[R[H_i]]$. By induction hypothesis $(A + N_{j-1}[R[S]])/N_{j-1}[R[S]]$ is nilpotent in $R[S]/N_{j-1}[R[S]]$; hence $A \subseteq N_j[R[S]]$. Therefore $\Sigma_{i \in I} N_j[R[H_i]] \subseteq N_j[R[S]]$. If j is a limit ordinal, then $N_j[R[H_1]] = \bigcup_{h<j} N_h[R[H_1]]$. Thus

$$\Sigma_{i \in I} N_j[R[H_i]] = \Sigma_{i \in I}(\bigcup_{h<j} N_h[R[H_i]]) = \bigcup_{h<j}(\Sigma_{i \in I}N_h[R[H_i]])$$

$$\subseteq \bigcup_{h<j} N_h[R[S]] = N_j[R[S]].$$

Since $N[R[H_i]] = N_\tau[R[H_i]]$ for some ordinal τ, eventually we get $\Sigma_{i \in I}N[R[H_i]] \subseteq N[R[S]]$.

So far we have obtained nilpotent ideals of R[S] as sums of nilpotent ideals of $R[H_i]$, $i \in I$. We now see that this is not always the case.

Lemma 3.15. Let A be a nonzero proper ideal of R[S]. Let $A_i = A \cap R[H_i]$, $i \in I$. Then the following statements hold:

(i) A_i is an ideal of $R[H_i]$ for every $i \in I$, and thus is a left ideal of R[S].

(ii) $A_i \simeq A_j$ for all $i, j \in I$.

(iii) If $A_i = 0$ for some $i \in I$, then $A_i = 0$ for all $i \in I$. In this case $AR[S] = 0$; hence A is nilpotent of index two.

Proof. (i) Since $A_i R[H_i] \subseteq AR[H_i] = A_i$, and symmetrically $R[H_i]A_i \subseteq A_i$, it follows that A_i is an ideal of $R[H_i]$ for every $i \in I$. By Lemma 3.13 each A_i is a left ideal of $R[S]$.

(ii) $A_i R[H_j] \subseteq A \cap R[H_j] = A_j$; hence $A_j \supseteq A_i e_j$. Symmetrically $A_i \supseteq A_j e_i = \bar{\theta}_{ij}(A_j)$. Thus

$$A_i \supseteq \bar{\theta}_{ji}(A_j) \supseteq \bar{\theta}_{ji}(A_i e_j) = A_i e_j e_i = A_i e_i = A_i.$$

Since $\bar{\theta}_{ij}$ are monomorphisms, we conclude that $A_i \cong A_j$.

(iii) Let $A_j = 0$ for some $j \in I$. By (ii) it is clear that $A_i = 0$ for all $i \in I$. By Lemma 3.12 (v) $R[H_i]$ is a left ideal of $R[S]$; hence $AR[H_i] \subseteq A \cap R[H_i] = 0$ for all $i \in I$. Then $AR[S] = A\Sigma_{i\in I}R[H_i] = \Sigma_{i\in I}AR[H_i] = 0$ by Lemma 3.12 (iii). In particular, $A^2 \subseteq AR[S] = 0$.

Definition. $W = \{ \sum_{i=1}^{n} a_i e_i \mid e_i \in E, \sum_{i=1}^{n} a_i = 0, a_i \in R,$ and $n \in Z^+\}$.

Lemma 3.16. Let L_1 be a left ideal of $R[H_1]$. Then $L_1 w$ is a nilpotent ideal of $R[S]$ for all $w \in W$, $w \neq 0$.

Proof. Let $x \in H_j$, and let $w = \sum_{i=1}^{n} a_i e_i \in W$. Since $wx = \sum_{i=1}^{n} a_i(e_i x) = (\sum_{i=1}^{n} a_i)x = 0$, then $wR[H_j] = 0$ for all $i \in I$. Hence $wR[S] = \Sigma_{j\in I}\, wR[H_j] = 0$. Thus $L_1 wR[S] \subseteq L_1 w$. By Lemma 3.12 (v) $R[S]L_1 = R[S](e_1 L_1) = R[H_1]L_1 = L_1$; so

$R[S]L_1w \subseteq L_1w$. Therefore, L_1w is a two-sided ideal of $R[S]$. Finally $(L_1w)^2 = L_1(wL_1)w = L_1 \cdot 0 \cdot w = 0$ implies that L_1w is nilpotent.

Proposition 3.17. Let $C = \{ \sum_{i_1=1}^{t_1} a_{i_1} e_{i_1} + \sum_{i_2=1}^{t_2} a_{\lambda_2 i_2} x_{\lambda_2 i_2} + \cdots + \sum_{i_n=1}^{t_n} a_{\lambda_n i_n} x_{\lambda_n i_n} \mid \sum_{i_1=1}^{t_1} a_{i_1} = \sum_{i_2=1}^{t_2} a_{\lambda_2 i_2} = \cdots = \sum_{i_n=1}^{t_n} a_{\lambda_n i_n} = 0;$ $x_{\lambda_j i_j} \in H_{i_j}$, $x_{\lambda_j m} e_k = x_{\lambda_j k}$, $m,k,i_j \in I$, t_j and $n \in Z^+$, $j = 1,2,\ldots,n\}$. Then C is a nilpotent ideal of $R[S]$ that contains every ideal A of $R[S]$ such that $AR[S] = 0$. Moreover, $C = R[H_1]W$.

Proof. Define ψ: $R[S] \to R[H_1]$ by $\psi(\alpha) = \alpha e_1$. By Lemma 3.12 (v) ψ is clearly an R-module epimorphism. Moreover,

$$\psi(\alpha\beta) = (\alpha\beta)e_1 = \alpha(\beta e_1) = \alpha(e_1\beta e_1) = (\alpha e_1)(\beta e_1) = \psi(\alpha)\psi(\beta),$$

and so ψ is also a ring epimorphism.

Let $\alpha \in \operatorname{Ker} \psi$. Write

$$\alpha = \sum_{j_1=1}^{t_1} a_{j_1} e_{j_1} + \sum_{j_2=1}^{t_2} a_{\lambda_2 j_2} x_{\lambda_2 j_2} + \cdots + \sum_{j_n=1}^{t_m} a_{\lambda_m j_m} x_{\lambda_m j_m},$$

where $x_{\lambda_i j_i} \in H_{j_i}$, $x_{\lambda_i n} e_k = x_{\lambda_i k}$; that is, the elements in the support of one summation belong to different subgroups, and all of them have the same image under any of the group isomorphisms θ_{nk} for all $n,k \in I$. Then

$$0 = \psi(\alpha) = \alpha e_1 = \sum_{j_1=1}^{t_1} a_{j_1}(e_{j_1}e_1) + \sum_{j_2=1}^{t_2} a_{\lambda_2 j_2}(x_{\lambda_2 j_2}e_1) + \cdots$$

$$+ \sum_{j_m=1}^{t_m} a_{\lambda_m j_m}(x_{\lambda_m j_m}e_1) = \Big(\sum_{j_1=1}^{t_1} a_{j_1}\Big)e_1 + \Big(\sum_{j_2=1}^{t_2} a_{\lambda_2 j_2}\Big)x_{\lambda_2 1} + \cdots$$

$$+ \Big(\sum_{j_m=1}^{t_m} a_{\lambda_m j_m}\Big)x_{\lambda_m 1}.$$

Hence $\sum_{j_1=1}^{t_1} a_{j_1} = \sum_{j_2=1}^{t_2} a_{\lambda_2 j_2} = \cdots = \sum_{j_m=1}^{t_m} a_{\lambda_m j_m} = 0$. It follows that $\alpha \in C$, and therefore Ker $\psi \subseteq C$.

On the other hand, it is clear that $\psi(C) = 0$, thus $C \subseteq$ Ker ψ, and we conclude that $C =$ Ker ψ is an ideal of $R[S]$.

From $\psi(R[H_1]W) = R[H_1](We_1) = R[H_1] \cdot 0 = 0$, we get $R[H_1]W \subseteq C$. Let $\alpha \in C$. By the definition of C we can write

$$\alpha = \sum_{j_1=1}^{t_1} a_{j_1}e_{j_1} + \sum_{j_2=1}^{t_2} a_{\lambda_2 j_2}x_{\lambda_2 j_2} + \cdots + \sum_{j_m=1}^{t_m} a_{\lambda_m j_m}x_{\lambda_m j_m}.$$

But $x_{\lambda_i 1}e_{j_i} = x_{\lambda_i j_i}$ implies that we can rewrite

$$\alpha = e_1 \sum_{j_1=1}^{t_1} a_{j_1}e_{j_1} + x_{\lambda_2 1} \sum_{j_2=1}^{t_2} a_{\lambda_2 j_2}e_{j_2} + \cdots +$$

$$+ x_{\lambda_m 1} \sum_{j_m=1}^{t_m} a_{\lambda_m j_m}e_{j_m},$$

and since $\sum_{j_i=1}^{t_i} a_{\lambda_i j_i} = 0$, $i = 1,\ldots,m$, then $\sum_{j_i=1}^{t_i} a_{\lambda_i j_i}e_{j_i} \in W$. Hence $\alpha \in R[H_1]W$, it follows that $C \subseteq R[H_1]W$. Therefore $C = R[H_1]W$. Whence $C^2 = R[H_1](WR[H_1])W = 0$ by Lemma 3.16. Thus C is nilpotent.

Finally, if A is an ideal of $R[S]$ such that $AR[S] = 0$, then $Ae_1 = 0$. Hence $A \subseteq \text{Ker } \psi = C$.

Proposition 3.18. Every element α of $R[S]$ can be uniquely expressed as $\alpha = \alpha_1 + \alpha_c$, where $\alpha_1 \in R[H_1]$ and $\alpha_c \in C$.

Proof. Let $\alpha \in R[S]$. As in the proof of Proposition 3.17 we can write

$$\alpha = \sum_{j_1=1}^{t_1} a_{j_1} e_{j_1} + \sum_{j_2=1}^{t_2} a_{\lambda_2 j_2} x_{\lambda_2 j_2} + \dots + \sum_{j_m=1}^{t_m} a_{\lambda_m j_m} x_{\lambda_m j_m},$$

where $x_{\lambda_i 1} e_{j_i} = x_{\lambda_i j_i}$. Set

$$d_1 = -\sum_{j_1=2}^{t_1} a_{j_1}, \quad d_2 = -\sum_{j_2=2}^{t_2} a_{\lambda_2 j_2}, \dots, \quad d_m = -\sum_{j_m=2}^{t_m} a_{\lambda_m j_m}.$$

Then we can rewrite α as $\alpha_1 + \alpha_c$, where

$$\alpha_1 = [(a_1 - d_1)e_1 + (a_{\lambda_2 1} - d_2)x_{\lambda_2 1} + \dots + (a_{\lambda_m 1} - d_m)x_{\lambda_m 1}]$$

and

$$\alpha_c = \sum_{j_1=2}^{t_1} a_{j_1} e_{j_1} + d_1 e_1 + \sum_{j_2=2}^{t_2} a_{\lambda_2 j_2} x_{\lambda_2 j_2} + d_2 x_{\lambda_2 1} + \dots +$$
$$+ \sum_{j_m=2}^{t_m} a_{\lambda_m j_m} x_{\lambda_m j_m} + d_m x_{\lambda_m 1}.$$

Clearly $\alpha_1 \in R[H_1]$, and since by construction $d_i + \sum_{j_i=1}^{t_i} a_{\lambda_i j_i} = 0$ for $i = 1, 2, \dots, m$, then $\alpha_c \in C$.

Suppose now that $\alpha = \alpha_1 + \alpha_c$ and also $\alpha = \beta_1 + \beta_c$, where $\alpha_1, \beta_1 \in R[H_1]$ and $\alpha_c, \beta_c \in C$. Since

$\alpha_1 - \beta_1 = \alpha_c - \beta_c \in R[H_1] \cap C = 0$, then $\alpha_1 = \beta_1$ and $\alpha_c = \beta_c$, which proves that the representation is unique.

We remark that Proposition 3.18 can be restated by saying that $R[S] = R[H_1] \oplus C$ is a direct sum of left ideals.

Theorem 3.19. Let R be a ring with identity, and let S be a right group. Then the nilpotent radical of R[S] is $N[R[S]] = \Sigma_{i\in I} N[R[H_i]] + C = N[R[H_1]]R[S] + C$.

Proof. By Propositions 3.14 and 3.17 it follows that

$$\Sigma_{i\in I} N[R[H_i]] + C \subseteq N[R[S]].$$

Hence $N[R[S]/C] = N[R[S]]/C \cong N[R[H_1]]$ by Lemma 3.18. Thus $N[R[S]] = N[R[H_1]] + C = N[R[H_1]]R[S] + C$.

Finally, by Proposition 3.12 (iii) $N[R[H_1]]R[S] = N[R[H_1]](\Sigma_{i\in I}R[H_i]) = \Sigma_{i\in I}N[R[H_1]](e_iR[H_i]) = \Sigma_{i\in I}N[R[H_i]]$.

3.4 Semigroup Rings Over Semigroups With Universally Minimal Ideals, and Their Nilradicals

An element u of a semigroup S is called a zeroid element of S if, for each element $s \in S$, there exist $x,y \in S$ such that $sx = ys = u$. It follows that if S has a zero element, this will be the only zeroid of S. Let A be an ideal of S. Then A is called locally minimal if it contains no proper subideal. A is called universally minimal in S if it is contained in every ideal of S. It is an easy consequence of these definitions that a semigroup S contains a universally minimal ideal A if and only if it contains a zeroid element, and then A consists of all the zeroids of S.

Clifford and Miller [4] have shown that the set U of zeroid elements of S is either vacuous or a subgroup and two-sided ideal of S. Moreover, the identity element z of U commutes with every element of S, and the mapping $\theta: S \to U: s \to zs(=sz)$ is an epimorphism. They define the core of S to be the subsemigroup $J = \{x \in S | xz = z\}$; thus J contains z as zero element. Finally the subsemigroup $U \cup J$ of S will be called the frame of S.

An easy example of a semigroup with universally minimal ideal is provided by the direct product of any group and any semigroup with zero.

Let R be a ring with identity, and let S be a semigroup without zero such that S contains a universally minimal ideal U. In this section we first show that R[S] is a ring direct sum of two known ideals. Using this decomposition, we find the general form of the nilpotent radical of R[S]. Finally we apply these results to semigroup rings over a subfamily of semigroups with universally minimal ideals, where the multiplication in the semigroups is known.

We begin by stating as a remark some facts taken from the main theorem in [4].

Remark 3.20. The following conditions hold in S:

(i) $\theta: S \to U: s \to sz$, is a semigroup epimorphism, where $z = 1_U$. Moreover, $zs = sz$ for all $s \in S$.

(ii) $S = \bigcup_{u \in U} J(u)$, where $J(u) = \theta^{-1}(u) = \{s \in S | sz=u=zs\}$.

(iii) $J(u) \cap U = u$ and $J(u) \cap J(v) = \phi$ for all $u,v \in U$, $u \neq v$.

(iv) $J(z) = J = \{s \in S | \ sz = zs = z\}$.

(v) $J(u)J(v) \subseteq J(uv)$ for all $u,v \in U$.

Lemma 3.21. Let $\bar{\theta}$: $R[S] \to R[U]$ be the natural extension of θ; that is, $\bar{\theta}(\Sigma a_i s_i) = \Sigma a_i \theta(s)$. Then $\bar{\theta}$ is a semigroup ring epimorphism.

Proof. By definition $\bar{\theta}$ is clearly an R-module homomorphism. Moreover, by Remark 3.20 (i) θ is an epimorphism. Since $zs = sz$ for all $s \in S$ and $z^2 = z$, it follows that θ is also a ring epimorphism.

Definition 3.22. Let $K(u) = \{ \sum_{i=1}^{n} a_i x_i | \sum_{i=1}^{n} a_i = 0$, $x_i \in J(u)$ and $n \in Z^+\}$ and set $K = \Sigma_{u \in U} K(u)$.

Lemma 3.23. K is an ideal of $R[S]$.

Proof. We show that $K = \operatorname{Ker} \bar{\theta}$. Let $\Sigma a_i x_i \in K(u)$. Since $\bar{\theta}(\Sigma a_i x_i) = \Sigma a_i \theta(x_i) = \Sigma a_i (x_i z) = (\Sigma a_i)u = 0$, then $\bar{\theta}(K(u)) = 0$. Thus, $\bar{\theta}(K) = 0$, and therefore $K \subseteq \operatorname{Ker} \bar{\theta}$.

On the other hand, let $\alpha \in \operatorname{Ker} \bar{\theta}$, and write

$$\alpha = \sum_i a_i u_i + \sum_j \sum_k b_{jk} x_{u_j k},$$

where $x_{u_j k} \in J(u_j) - \{u_j\}$ for all k. Then

$$0 = \bar{\theta}(\alpha) = \sum_i a_i u_i + \sum_j \sum_k b_{jk}(x_{u_j k} z) = \sum_i a_i u_i + \sum_j (\sum_k b_{jk}) u_j.$$

After renaming conveniently the subindexes, we can write $0 = \bar{\theta}(\alpha) = \sum_t (a_t + \sum_k b_{tk}) u_t$; hence $a_t + \sum_k b_{tk} = 0$ for each t. Thus

$$\alpha = \sum_t (a_t u_t + \sum_k b_{tk} x_{u_t k}) \in \sum_t K(u_t) \subseteq K.$$

Therefore Ker $\bar{\theta} = K$.

We are now in the position to show that the semigroup ring of a semigroup having a universally minimal ideal is completely determined by the group ring of the group of zeroids and the ideal K.

Theorem 3.24. $R[S] = R[U] \oplus K$ (ring direct sum).

Proof. It follows from the definition of $K(u)$ and (ii) of Remark 3.20 that $R[U] \cap K(u) = 0$ for all $u \in U$. Hence $R[U] \cap K = 0$.

By Lemma 3.23 K is an ideal of R[S]. Since U is an ideal of S, we get readily that R[U] is also an ideal of R[S]. As K is the kernel of the epimorphism $\bar{\theta}$, it follows that $R[S]/K \simeq R[U]$. Thus $R[S] = R[U] + K$.

Corollary 3.25. If the semigroup S coincides with its frame, then $R[S] = R[U] \oplus K(z)$.

Proof. Since $S = U \cup J$, then Ker $\bar{\theta} = K(z)$ and the result follows from Theorem 3.24.

Proposition 3.26. The nilpotent radical of R[S] is $N[R[S]] = N[R[U]] \oplus N[K]$. In particular, if $S = U \cup J$, then $N[R[S]] = N[R[U]] \oplus N[K(z)]$.

Proof. This follows directly from Theorem 3.24 and Corollary 3.25.

Corollary 3.27. Let $S = U \cup J$, where J is a right zero semigroup with zero, and let R be an integral domain. Then $N[R[S]] = N[R[U]] \oplus K(z)$.

Proof. Since $K(z)\cdot z = z\cdot K(z) = 0$, then $K(z)$ is an ideal of $R[J]$. By Theorem 3.5 $K(z)$ is nilpotent; hence the result follows by Proposition 3.26.

So far we have seen that the nilpotent radical of $R[S]$ is determined by the nilpotent radicals of $R[U]$ and K. If we want to find out what the nilpotent radical of K looks like, we need to know how the multiplication among the elements of the sets $J(u)$ is defined. The classification for the possible infinite semigroups with a given frame is an open problem; next we sketch a simple construction due to Clifford and Miller [4] that shows there exist an infinite number of such semigroups, and we find their nilpotent radical.

Let the group U and the semigroup J with zero be given. By identifying the identity of U with the zero of J and defining $xu = ux = u$ for all $u \in U$ and $x \in J$, we get a semigroup S_0 whose frame coincide with itself. This S_0 is a special case of the following type of semigroup.

Associate with each element $u \in U$ a set $J(u)$ in any way subject to the conditions (iii) and (v) of the Remark 3.20. For each $u \in U$, determine a right and a left representation of J by single valued mappings of the set $J(u)$ into itself such that (a) left and right mappings commute with each other, (b) the element $z \in J$ maps every element of $J(u)$ into u, and (c) all the mappings leave fixed the element $u \in J(u)$. The multiplication in the class sum S of all $J(u)$ is defined as follows. Within $J(z)$ it is defined as originally given in J. For $k \in J$ and $a \in J(u)$ we define $ak(ka)$ to be the image of a under the mapping corresponding to k in the right (left) representation already determined. Finally for $x_u \in J(u)$ and $x_v \in J(v)$ where $u \neq z$ and $v \neq z$, we define $x_u x_v = uv$, where uv is the product of u and v as originally given in U. Then each $S = \bigcup_{u \in U} J(u)$ is a semigroup with frame $S_0 = U \cup J$.

Lemma 3.28. The sets $K(u)$ are nilpotent ideals of index two in $R[S]$ for all $u \neq z$.

Proof. Let $\alpha = \sum_i a_i x_i \in K(u)$, let $\rho = \sum_k c_k x_k \in R[J[(v)]$ where $u \neq z$ and $v \neq z$, and let $\delta = \sum_t d_t y_t \in R[J(z)]$. Since $\sum_i a_i = 0$, then $\alpha\rho = (\sum_i a_i x_i)(\sum_k c_k a_k) = \sum_k \sum_i a_i c_k (x_i a_k) = \sum_k (\sum_i a_i) c_k (uv) = 0$. Hence $K(u)R[J(v)] = 0$ for all $u, v \in U$ with $u \neq z$ and $v \neq z$. In particular, $K_u^2 \subseteq K_u R[J(u)] = 0$. On the other hand, $\alpha\delta = (\sum_i a_i x_i)(\sum_t d_t y_t) = \sum_t \sum_i a_i d_t (x_i y_t)$. Since $x_i y_t \in J(u)J(z) \subseteq J(u)$ and since $\sum_t \sum_i a_i d_t = (\sum_i a_i)(\sum_t d_t) = 0$, then $\alpha\delta \in K(u)$. Therefore $K(u)\ R[J(z)] \subseteq K(u)$. Moreover, from $S = \cup J(u)$ we get $R[S] = \Sigma_{u \in U} R[J(u)]$. Hence

$$K(u)R[S] = K(u)[\sum_{v \in U - \{z\}} R[J(v)] + R[J(z)]] = K(u)R[J(z)] \subseteq K(u).$$

Symmetrically $R[S]K(u) \subseteq K(u)$, and the result follows.

Theorem 3.29. The nilpotent radical of R[S] is $N[R[S]] = N[R[U]] \oplus (\Sigma_{u\in U-\{z\}}K(u) + N[K(z)])$.

Proof. By Proposition 3.26 $N[R[S]] = N[R[U]] \oplus N[K]$. But $N[K] = N[\Sigma_{u\in U}K(u)]$, and since K(u) is a nilpotent ideal of index two for $u \neq z$ by Lemma 3.28, then $N[K] = N[\Sigma_{u\in U}K(u)] = \Sigma_{u\in U-\{z\}}K(u) + (N[R[J]] \cap K(z) = \Sigma_{u\in U-\{z\}}K(u) + N[K(z)]$.

CHAPTER IV
SEMIPRIME SEMIGROUP RINGS

In commutative semigroup rings Gilmer and Parker [10] have proven the following result. Let R be a commutative ring with identity, and let S be an abelian semigroup. If either the characteristic of R is zero and S is torsion free or else the characteristic of R is n and S is p-torsion free for each prime divisor p of n, then N[S] is the nilpotent radical of R[S], where N is the nilpotent radical of R. It follows clearly that under the conditions of this result R[S] is semiprime if and only if R is semiprime.

In this chapter we are mainly concerned with the problem of determining conditions on R or S that will give rise to semiprime (twisted) semigroup rings.

We have already seen in Corollary 3.7 that if E is a right (left) zero semigroup and R is a commutative ring with identity, then R[E] is semiprime if and only if R is an integral domain and $|E| = 1$. As a direct consequence of Theorem 3.19 we have that if S is a right (left) group with more than one idempotent (that is, S is not a group) and R is a ring with identity, then R[S] is never semiprime. Finally, if S is a semigroup with universally minimal ideal U and R is a ring with identity, then it follows from Proposition 3.26 that R[S] is semiprime if R[U] and K are semiprime, where K is the kernel ideal of the canonical epimorphism from R[S] into R[U].

Now we let R be a ring. In Theorem 4.5 we prove that if S is a semigroup satisfying a suitable condition, then the twisted semigroup ring $R^t[S]$ is semiprime if and only if R is semiprime.

In section two we consider two types of semilattices. The first is a semilattice P_0 with a zero element and in which all the nonzero idempotents are primitive. The second type of semilattice P_1 has also zero element and satisfies the following condition: if F is a subset of P_1, then either $g\cdot l\cdot b\{x|x \in F\}$ belongs to F or else there exist x,y in F such that $g\cdot l\cdot b\{x,y\}$ is in $P_1 - F$.

Next, we consider two families of semigroups which admit relative inverses. We say that a semigroup S belongs to the family $\mathcal{S}$ if (i) S admits relative inveres, (ii) the idempotents of S commute, and (iii) the semilattice of subgroups S_α of S is isomorphic to P_0. Then in Theorem 4.11 we decompose R[S] as an internal weak direct sum of ideals which are completely determined once $R[S_\alpha]$ is known for each $\alpha \in P_0$. This allows us to characterize the nilpotent and Jacobson radicals of R[S]. Further, if we let F be a field and use results of group ring theory, we can give necessary conditions for F[S] to be semisimple and regular in Corollaries 4.15, 4.16 and 4.17.

Passman has related semiprime group rings with their center in the following two results [22, Theorems 2.12 and 2.13].

(A) Let $K[G]$ be a group ring over a field of characteristic 0. Then (i) $K[G]$ is semiprime, (ii) $Z[K[G]]$ is semiprime, and (iii) $Z[K[G]]$ is semisimple.

(B) Let $K[G]$ be a group ring over a field of characteristic $p > 0$. Then the following are equivalent.

(i) $K[G]$ is semiprime.
(ii) $Z[K[G]]$ is semiprime.
(iii) $Z[K[G]]$ is semisimple.
(iv) G has no finite normal subgroups H with $p \mid |H|$.
(v) $\Delta(G)$ is a p'-group, where $\Delta(G) = \{x \in G \mid$ the index of the centralizer of x in G is finite$\}$.

In Corollary 4.18 we apply (A) and (B) to the group rings $F[S_\alpha]$, $\alpha \in P_0$, to obtain a similar result between $F[S]$ and $Z[F[S]]$.

Finally, we say that a semigroup S belongs to the family τ if (i) S admits relative inverses, (ii) the idempotents of S commute, and (iii) the semilattice of subgroups S_α of S is of type P_1. Then in Theorem 4.21 we prove that $R[S]$ is semiprime if $R[S_\alpha]$ is semiprime for all $\alpha \in P_1$.

4.1 Semiprime Twisted Semigroup Rings

In this section we first show that a twisted semigroup ring $R^t[S]$ is semiprime if and only if the ring R is semiprime, provided the semigroup S satisfies a suitable condition. From this we are able to determine the nilpotent radical of this semigroup ring.

The following definitions are due to Banaschewski [2].

Definition 4.1. Let X be a subset of a semigroup S. An element x of X is called an extremity of X is for all positive integer n, $x^n = s_1s_2\ldots s_n$, $s_i \in X$, implies $s_i = x$ for all i.

Definition 4.2. A semigroup S is said to satisfy condition (E) if every nonempty subset of S has an extremity.

It is clear that any totally ordered semigroup S satisfies condition (E), since the greatest and the least element of any finite subset X of S, are extremities of X. Moreover, if a semigroup S satisfies condition (E) and $x^n = y^n$ for $x,y \in S$, then $x = y$ for $n = 1,2,\ldots$; that is, S is power cancellative. Banaschewski proved also in [2], that any cancellative, power cancellative abelian semigroup satisfies condition (E).

We might think of the possibility of defining a weaker condition than (E) on S, both conditions related in the same way as the p-torsion free condition is related to the torsion free one. Namely, we would define a p-extremity of a subset X of S as an element a such that $a^{p^n} = s_1s_2\ldots s_{p^n}$, $s_i \in X$, implies that $a = s_1 = s_2 = \ldots = s_{p^n}$. Then we could set that a semigroup S satisfies condition (E_p) if any nonempty finite subset X of S has a p-extremity. This would allow us to differentiate between twisted semigroup rings over rings of finite characteristic from those whose rings have characteristic 0, as Gilmer and Parker [10] did in commutative semigroup rings.

Now assume that S satisfies condition (E_p), and let a be a p-extremity of a finite subset X of S. Let $a^n = s_1 s_2 \ldots s_n$, $s_i \in X$, and choose $m \in Z^+$ such that $p^m > n$. Then $a^{p^m} = a^n a^{p^m - n} = s_1 s_2 \ldots s_n\ a^{p^m - n}$. Hence condition (E_p) implies that $a = s_1 = s_2 = \ldots = s_n$; that is, a is an extremity. This shows that (E) and (E_p) are equivalent conditions in S.

We begin by emphasizing a direct consequence of Definition 4.1.

Lemma 4.3. Let S be a semigroup satisfying condition (E), and let X be a nonvoid finite subset of S. If a is an extremity of X and n is a positive integer, then a^n is an extremity of $X^n = \{x_1 x_2 \ldots x_n | x_i \in X\}$.

Proof. Let $(a^n)^m = y_1 y_2 \ldots y_m$, where $m \in Z^+$ and each $y_i = x_{i1} x_{i2} \ldots x_{in}$ is an element of X^n for $1 \le i \le m$. Then $(a^n)^m = a^{nm} = (x_{11} x_{12} \ldots x_{1n}) \ldots (x_{m1} x_{m2} \ldots x_{mn})$. But a is an extremity of X; hence $a = x_{ij}$ for all i and j. Therefore $a^n = y_1 = \ldots = y_m$.

Proposition 4.4. Let R be a ring, and let S be a semigroup. If the twisted semigroup ring $R^t[S]$ is semiprime, then R is semiprime.

Proof. Assume that $R^t[S]$ is semiprime. By way of contradiction, suppose that there exists a nonzero nilpotent ideal A of R. By section 1.5 $A^t[S]$ is a nonzero ideal of $R^t[S]$. Let m be the index of nilpotency of A, and let $\alpha = \sum_{i=1}^{k} a_i s_i$

be a nonzero element of $A^t[S]$. Then $\alpha^m = (\sum_{i=1}^{k} a_is_i)^m = \sum(a_{i_1}s_{i_1})(a_{i_2}s_{i_2})\cdots(a_{i_m}s_{i_m})$ with L = 1,2,...,k and $(i_1,i_2,\dots i_m) \in L\times\dots\times L = L^m$. But $(a_{i_1}s_{i_1})(a_{i_2}s_{i_2})\cdots(a_{i_m}s_{i_m}) = a_{i_1}a_{i_2}\gamma(s_{i_1},s_{i_2})a_{i_3}\cdots a_{i_m}\gamma(s_{i_1}s_{i_2}\cdots s_{i_{m-1}},s_m)\, s_1s_2\cdots s_m = 0$, since $a_{i_1}a_{i_2}\gamma(s_{i_1}s_{i_2})a_{i_3}\cdots a_{i_m}\gamma(s_{i_1}s_{i_2}\cdots s_{i_{m-1}},s_{i_m}) \in A^m = 0$. Hence $\alpha^m = 0$. Therefore $A^t[S]$ is a nilpotent ideal of $R^t[S]$, which contradicts our assumption; thus R is semiprime.

Theorem 4.5. Let R be a ring, and let S be a semigroup satisfying condition (E). Then the following conditions are equivalent:

(i) The twisted semigroup ring $R^t[S]$ is semiprime.

(ii) R is semiprime.

Proof. If $R^t[S]$ is semiprime, then by Proposition 4.4 R is semiprime.

Conversely, assume that R is semiprime. By way of contradiction, suppose that $R^t[S]$ contains a nonzero nilpotent ideal B. Let h be the index of nilpotency of B, and let $\beta = \sum_{i=1}^{r} b_is_i$ be a nonzero element of B. Then

$$0 = \beta^h = (\sum_{i=1}^{r} b_is_i)^h = \sum_{i=1}^{r} (b_is_i)^r + \sum(b_{i_1}s_{i_1})\cdots(b_{i_n}s_{i_n})$$

with $M = \{1,2,\dots,r\}$ and $(i_1,i_2,\dots,i_h) \in \{x \in M^h \mid$ not all coordinates of x are equal$\}$. Now supp β is a finite subset of S, and since by hypothesis, condition (E) holds in S, then there exists s_j extremity in supp β. Thus $s_j^h \neq s_{i_1}s_{i_2}\cdots s_{i_n}$

whenever $i_k \neq j$ for some $k = 1,2,\ldots,h$. It follows that $0 = (b_j s_j)^h = b_j^2 \gamma(s_j s_j) b_j \ \ldots\ \gamma(s_j^{h-1}, s_j) s_j^h = b_j^h c s_j^h$, where $c = \gamma(s_j, s_j)\gamma(s_j^2, s_j)\ \ldots\ \gamma(s_j^{h-1}, s_j)$. Hence $b_j^h c = 0$. Since the images of γ belong to the units of R, then c is a unit. Therefore $b_j^h = 0$; that is, b_j is a nilpotent element of R.

But B is an ideal; so the set $D = \{(x\gamma(s_j^2, s_j)^{-1} s_j^2)\beta + \beta(y\gamma(s_j, s_j^2)^{-1} s_j^2) + (\gamma(s_j^2, s_j)^{-1} s_j^2) n\beta + \sum_{i=1}^{m} (x_i \gamma(s_j, s_j)^{-1} s_j)\beta(y_i \gamma(s_j^2, s_j)^{-1} s_j) \mid x, y, x_i, y_i \in R,\ n \in Z,\ m \in Z^+\}$ is contained in B. Since s_j is an extremity of supp β, it follows from Lemma 4.3 that s_j^3 is an extremity of the support of any element of D. Hence all the elements of the principal ideal

$$(b_j) = \{xb_j + b_j y + nb_j + \sum_{i=1}^{m} x_i b_j y_i \mid x, y, x_i y_i \in R,\ n \in Z,\ m \in Z^+\}$$

of R are coefficients of s_j^3 in elements of B having s_j^3 as an extremity of their support. By the argument in the previous paragraph, the h-th power of each of these elements is zero. Therefore (b_j) is a nonzero nilpotent ideal of R, which contradicts our hypothesis.

The preceding result allows us to determine the nilpotent radical of $R^t[S]$.

Theorem 4.6. Let R be a ring, and let S be a semigroup satisfying condition (E). Then the nilpotent radical of the twisted semigroup ring $R^t[S]$ is $N[R^t[S]] = N^t[S]$, where N is the nilpotent radical of R.

Proof. Let $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_j \subseteq \cdots$ and $N_0[R^t[S]] \subseteq N_1[R^t[S]] \subseteq \ldots$, be the chains of ideals used in the definition of the nilpotent radical of R and $R^t[S]$, respectively. We show that $N^t[S] \subseteq N[R^t[S]]$ by transfinite induction. We have seen in the proof of Proposition 4.4 that $B^t[S]$ is a nilpotent ideal of $R^t[S]$ for each nilpotent ideal B of R. Hence $N_o^t[S] \subseteq N_0[R^t[S]]$. This shows the first step of the induction argument.

Assume that $N_i^t[S] \subseteq N_i[R^t[S]]$ for all the ordinals $i < j$. Suppose j is not a limit ordinal and let A/N_{j-1} be a nilpotent ideal of R/N_{j-1} with index of nilpotency m. Let $f = \sum_{r=1}^{k} a_r s_r \in A^t[S]$. Then $f^m = (\sum_{r=1}^{k} a_r s_r)^m = \Sigma(a_{r_1}s_{r_1})(a_{r_2}s_{r_2})\cdots(a_{r_m}s_{r_m})$ with $X = \{1,2,\ldots,k\}$ and $(r_1,r_2,\ldots,r_m) \in X \times \ldots \times X = X^m$. But

$$(a_{r_1}s_{r_1})(a_{r_2}s_{r_2})\cdots(a_{r_m}s_{r_m}) = a_{r_1}a_{r_2}\gamma(s_{r_1},s_{r_2})a_{r_3}\cdots a_{r_m}$$
$$\gamma(s_{r_1}\cdots s_{r_{m-1}},s_{r_m})s_{r_1}\cdots s_{r_m} \in N_i^t[S] .$$

Since $a_{r_1}a_{r_2}\gamma(s_{r_1},s_{r_2})a_{r_3}\cdots a_{r_m}\gamma(s_{r_1}\cdots s_{r_{m-1}},s_{r_m}) \in A^m = 0$. Thus $(A/N_{j-1})^t[S]$ is a nilpotent in $(R/N_{j-1})^t[S]$; hence $A^t[S]/N_{j-1}{}^t[S]$ is a nilpotent ideal of $R^t[S]/N_{j-1}{}^t[S]$. By induction hypothesis $A^t[S] + N_{j-1}[R^t[S]]/N_{j-1}[R^t[S]]$ is nilpotent in $R^t[S]/N_{j-1}\ R^t[S]]$; so $A^t[S] \subseteq N_j[R^t[S]]$. Therefore $N_j^t[S] \subseteq N_j[R^t[S]]$. If j is a limit ordinal, then $N_j = \bigcup_{i<j} N_i$. Since by induction hypothesis $N_i^t[S] \subseteq N[R^t[S]]$

for all $i < j$, then

$$N_j^t[S] = \bigcup_{i<j} N_i^t[S] \subseteq \bigcup_{i<j} N_i[R^t[S]] = N_j[R^t[S]].$$

Finally, since $N = N_\tau$ for some ordinal τ, we have that $N^t[S] \subseteq N[R^t[S]$.

Conversely, since R/N is a semiprime ring, it follows from Theorem 4.5 that $(R/N)^t[S]$ is semiprime; thus $R^t[S]/N^t[S]$ is semiprime. Since $N[R^t[S]]/N^t[S] \subseteq N[(R/N)^t[S]] = 0$, then $N[R^t[S]] \subseteq N^t[S]$.

Proposition 4.7. Let R be a ring with identity, and let S be a semigroup with identity in which condition (E) holds. Denote by $Z[R^t[S]]$ the center of $R^t[S]$. If R is semiprime, then $Z[R^t[S]]$ is semiprime.

Proof. By Theorem 4.5 and the hypothesis, it follows that $R^t[S]$ is semiprime. Suppose that there exists an ideal A of $Z[R^t[S]]$ such that $A^2 = 0$. Let $\alpha \in A$, then $\alpha^2 = 0$. Moreover, since α is in the center, $\alpha R^t[S]$ is a two-sided ideal of $R^t[S]$. Now $(\alpha R^t[S])^2 = \alpha^2 R^t[S] = 0$. Since $R^t[S]$ is semiprime, it follows that $\alpha R^t[S] = 0$; hence $\alpha = 0$, and consequently $Z[R^t[S]]$ is semiprime.

4.2 On Semigroup Rings Over Semigroups Admitting Relative Inverses

In this section we apply the results of section 3.4 to semigroups of a type first discussed by Clifford [3] and largely studied since then [5], [6], [19].

A semigroup S is said to admit relative inverses if, for any element s in S, there exist elements e and s' in S such that es = se = s and ss' = s's = e. By a semilattice we mean a commutative semigroup all of whose elements are idempotents. Let α and β belong to the semilattice P; then if $\alpha\beta = \beta$ we write $\alpha \geq \beta$. This defines a partial order on P with the property that any two elements have a greatest lower bound, namely its product.

We are interested in semigroups admitting relative inverses and such that every pair of idempotents commute.

Clifford's main result [3] on these semigroups may then be stated thusly:

Every semigroup which admits relative inverses and such that every pair of its idempotents commute, is isomorphic with a semigroup S constructed as follows.

Let P be any semilattice, and to each α in P assign a group S_α such that no two of them have an element in common. To each pair of elements $\alpha > \beta$ of P assign a homomorphism $\phi_{\alpha\beta}: S_\alpha \to S_\beta$ such that if $\alpha > \beta > \gamma$, then $\phi_{\alpha\beta}\phi_{\beta\gamma}=\phi_{\alpha\gamma}$. We call this a transitivity condition on the homomorphisms. Let $\phi_{\alpha\alpha}$ be the identity automorphism of S_α. Let S be the class sum of the groups S_α, and define the product of any two elements a_α, b_β of S($a_\alpha \in S_\alpha$ and $b_\beta \in S_\beta$) by $a_\alpha b_\beta = (a_\alpha\phi_{\alpha\gamma})(b_\beta\phi_{\beta\gamma})$, where $\alpha\beta = \gamma$ is the product of α and β in P.

Conversely, any semigroup S constructed in this fashion admits relative inverses, and every idempotent element of S is in the center of S.

The semilattice P, the groups S_α, and the homomorphisms $\phi_{\alpha\beta}$ together constitute a complete set of invariants of S.

The only result that we know on semigroup algebras over this type of semigroup is due to Munn [18]. He proved that under suitable conditions on the characteristic of the field and the order of the subgroups, the semigroup algebra of a finite semigroup which admits relative inverses is semisimple if and only if the idempotents commute.

Let P_0 be a semilattice with zero elements w such that every element of $P_0 - \{w\}$ is primitive; that is, for all $\alpha, \beta \in P_0 - \{w\}$, $\alpha\beta = w$. Denote by $\mathcal{S}$ the family of all semigroups S with the following properties: (i) S admits relative inverses; (ii) the idempotents in S commute; and (iii) S has P_0 as the semilattice of subgroups.

Let $S \in \mathcal{S}$, and let $\{S_\alpha\}_{\alpha \in P_0}$ and $\{\theta_{\alpha w}\colon S_\alpha \to S_w \mid \alpha \in P_0\}$ stand for the sets of semigroups and homomorphisms of S respectively. Finally, let R be a ring not necessarily with 1. Our first goal in this section is to characterize R[S] as a weak direct sum of ideals which are completely determined once the $R[S_\alpha]$ are known.

We begin by relating the semigroup S with the semigroups having a universally minimal ideal.

Lemma 4.8. Let $S \in \mathcal{S}$. The following conditions hold in S.

(i) S is a semigroup with universally minimal ideal S_w.

(ii) The frame of S is $S_w \cup J$, where $J = \cup_{\alpha \in P_0} \operatorname{Ker} \theta_{\alpha w}$.

(iii) The classes $J(u)$ are defined by $J(u) = \cup_{\alpha\epsilon P_0}\theta^{-1}_{\alpha w}(u)$ for each $u \in S_w$.

(iv) Let $S_\alpha(u) = S_\alpha \cap J(u)$; then $S_\alpha(u)S_\beta(v) = uv$, and $S_\alpha(u)S_\alpha(v) \subseteq S_\alpha(uv)$.

Proof. (i) Let $a_\alpha \in S_\alpha$ and $u \in S_w$. Then $a_\alpha u = \theta_{\alpha w}(a_\alpha)\theta_{ww}(u) \in S_w$; that is, $S_\alpha S_w \subseteq S_w$ for all $\alpha \in P_0$. Hence $SS_w \subseteq S_w$, and symmetrically $S_wS \subseteq S_w$. This shows that S_w is an ideal of S. Moreover, S_w is a group and therefore is a simple semigroup. Finally, since $xy \in S_w$ for $x \in S_\alpha$, $y \in S_\beta$, where α and β are distincts and $\alpha \neq w \neq \beta$, it follows that S_w is universally minimal.

(ii) By definition $J = \{s \in S | se_w = e_w\}$. Let $x \in J \cap S_\alpha$ for some α in P_0. Then $e_w = xe_w = \theta_{\alpha w}(x)e_w$ implies that $\theta_{\alpha w}(x) = e_w$; hence $x \in \text{Ker } \theta_{\alpha w}$. On the other hand, if $y \in \text{Ker } \theta_{\alpha w}$, then $ye_w = \theta_{\alpha w}(y)e_w = e_w$ shows that $y \in J$, and the result follows.

(iii) By definition $J(u) = \{s \in S | se_w = u\}$. Let $s \in S_\alpha \cap J(u)$. Then $u = se_w = \theta_{\alpha w}(s)e_w$ implies that $\theta_{\alpha w}(s) = u$; that is $s \in \theta^{-1}_{\alpha w}(u)$. Moreover, if $t \in \theta^{-1}_{\alpha w}(u)$, then $te_w = \theta_{\alpha w}(t)e_w = u$ shows that $t \in J(u)$. Therefore $J(u) = \cup_{\alpha\epsilon P_0}\theta^{-1}_{\alpha u}(u)$.

(iv) Let $x,z \in S_\alpha(u)$ and $y \in S_\beta(v)$ with $\alpha,\beta \in P$ and $u,v \in S_w$. Then $\theta_{\alpha w}(x) = u$ and $\theta_{\beta w}(y) = v$. So for $\alpha \neq \beta$ $xy = \theta_{\alpha w}(x)\theta_{\beta w}(y) = uv$ implies that $S_\alpha(u)S_\beta(v) = uv$. Moreover, $\theta_{\alpha w}(xz) = \theta_{\alpha w}(x)\theta_{\alpha w}(z) = uv$; so $xz \in S_\alpha \cap J(uv) = S_\alpha(uv)$. Thus $S_\alpha(u)S_\alpha(v) \subseteq S_\alpha(uv)$.

Recalling definition 3.22, we let $K(u) = \{ \sum_{i=1}^{n} a_i x_i \mid \sum_{i=1}^{n} a_i = 0$ and $x_i \in J(u)\}$ and $K = \Sigma_{u \in S_w} K(u)$.

Now we can apply the results of section 3.4 to obtain the following result.

Proposition 4.9. $R[S] = R[S_w] \oplus K$ as a ring direct sum.

Proof. This follows directly from Lemma 4.8 and Theorem 3.24.

Define $C_\alpha(u) = \{ \sum_{i=1}^{m} r_i(x_i - u) \mid r_i \in R,\ x_i \in S_\alpha(u)$ and $m \in Z^+\}$ and $C_\alpha = \Sigma_{u \in S_w} C_\alpha(u)$.

Lemma 4.10. C_α is an ideal of $R[S]$ for each $\alpha \in P_0$.

Proof. First notice that $C_w = 0$, since $S_w(u) = u$ for each $u \in S_w$. Assume $\alpha \neq w$, $\alpha \in P_0$. By definition it is clear that $C_\alpha \subseteq K$. Hence $C_\alpha R[S_w] \subseteq KR[S_w] = 0$ and $R[S_w]C_\alpha \subseteq R[S_w]K = 0$. To show that C_α is closed under multiplication, it is enough to prove that the product of the generators of C_α is in C_α. Let $x-u$ and $y-v$ belong to $C_\alpha(u)$ and $C_\alpha(v)$, respectively. Then $(x-u)(y-v) = xy-xv-uy+uv = xy-\theta_{\alpha w}(x)v-u\theta_{\beta w}(y)+uv = xy-2uv+uv = xy-uv$. By Lemma 4.8 (iv) $xy \in S_\alpha(uv)$; hence $xy-uv \in C_\alpha(uv)$. Therefore, $C_\alpha(u)C_\alpha(v) \subseteq C_\alpha(uv)$ for all $u,v \in S_w$. It follows that $C_\alpha C_\alpha \subseteq C_\alpha$.

Now let $x-v \in C_\alpha(v)$, and let $\alpha_u = \sum_i a_i x_i + \sum_i b_j y_j$ belong to K_u, where $x_i \in S_\alpha(u)$ and $y_j \in J(u)-S_\alpha(u)$. Then

$\alpha_u = \sum_i a_i(x_i-u) + \sum_j b_j y_j + (\sum_i a_i)u$. After remaining conveniently we can write $\alpha_u = \sum_i a_i(x_i-u) + \sum_k c_k z_k$, where each $z_k \in S_\beta(u)$ for some $\beta \neq \alpha$, $\beta \in P_0$. Then $z_k(x-v) = z_k x - z_k v = \theta_{\beta w}(z_k)\theta_{\alpha w}(x) - \theta_{\beta w}(z_k)v = uv-uv = 0$, and $(x_i-u)(x-v) = x_i x - x_i v - ux + uv = x_i x - \theta_{\alpha w}(x_i)v - u\theta_{\alpha w}(x) + uv = x_i x - 2uv + uv = x_i x - uv \in C_\alpha(uv)$ since $x_i x \in S_\alpha(u)S_\alpha(v) \in S_\alpha(uv)$ by lemma 4.8 (iv). Hence $\alpha_u(x-v) = \sum_i a_i(x_i-u)(x-v) + \sum_k c_k z_k(x-v) = \sum_i a_i(x_i x - uv) \in C_\alpha(uv)$. Therefore $K(u)C_\alpha(v) \subseteq C_\alpha(uv)$. It follows now that $KC_\alpha = (\sum_u K(u))(\sum_v C_\alpha(v)) = \sum\sum_{vu} K_u C_\alpha(v) \subseteq \sum_v C_\alpha(uv) = C_\alpha$. Symmetrically $C_\alpha K \subseteq C_\alpha$. Thus C_α is an ideal of K. Then by Proposition 4.9 C_α is an ideal of R[S].

Theorem 4.11. The ideal $K = \sum^w_{\alpha\in P_0} C_\alpha$ is the internal weak direct sum of the ideals C_α. Hence $R[S] = R[S_w] \oplus \sum_{\alpha\in P_0} C_\alpha$.

Proof. Let $f \in K$. From $K = \sum_{u\in S_w} K(u)$ and $(\sum_{u\neq v} K(u)) \cap K(v) = 0$, it follows that f can be written uniquely as $f = \sum_{u\in S_w} f_u$ with $f_u \in K(u)$. Let $f_u = \sum_{j_1=1}^{n_1} a_{j_1} x_{\alpha_1 j_1} + \cdots + \sum_{j_k=1}^{n_k} a_{j_k} x_{\alpha_k j_k}$, where $x_{\alpha_i j_i} \in S_{\alpha_i}(u)$. Since $f_u \in K(u)$, we may write $f_u = \sum_{j_1=1}^{n_1} a_{j_1}(x_{\alpha_1 j_1}-u) + \cdots + \sum_{jk}^{n_k} a_{j_k}(x_{\alpha_{j_k}}-u)$. Suppose that f_u can be expressed in a different way; say $f_u = \sum_{t_1=1}^{m_1} b_{t_1}(y_{\alpha_1 t_1}-u) + \cdots + \sum_{t_k=1}^{m_k} b_{t_k}(y_{\alpha_k t_k}-u)$ with $y_{\alpha_i t_i} \in S_{\alpha_i}(u)$. Then

$$(\sum_{j_1=1}^{n_1} a_{j_1}(x_{\alpha_1 j_1}-u) - \sum_{t_1=1}^{m_1} b_{t_1}(y_{\alpha_1 t_1}-u) + \cdots +$$

$$(\sum_{j_k=1}^{n_k} a_{j_k}(x_{\alpha_k j_k}-u) - \sum_{t_k=1}^{n_k} b_{t_k}(y_{\alpha_k t_k}-u)) = 0.$$

But the support of any two sets of terms in parenthesis are disjoint; hence each term inside of a parenthesis must be 0. Then $\sum_{j_i=1}^{n_i} a_{j_i}(x_{\alpha_i j_i}-u) = \sum_{t_i=1}^{m_i} b_{t_i}(y_{\alpha_i t_i}-u)$ in $C_{\alpha_i}(u)$ for all i. It follows that, except by possibly for order, the elements are identical. This shows that each f_u has a unique representation in $\Sigma_{\alpha\in P_0} C_\alpha(u)$; hence f can be uniquely expressed in $\sum_{u\alpha}\sum C_\alpha(u) = \sum_\alpha C_\alpha$.

Therefore, K is the internal weak direct sum of its ideals C_α. By Proposition 4.9 we conclude that $R[S] = R[S_w] \oplus \Sigma^w_{\alpha\in P_0} C_\alpha$.

Lemma 4.12. For each $\alpha \in P_0 - \{w\}$, the ideal C_α is isomorphic as an algebra with $R[S_\alpha]$.

Proof. Let $f \in R[S_\alpha]$ with $\alpha \neq w$. Since $S_\alpha = \dot{\cup}_{u\in S_w} S_\alpha(u)$, we can write $f = \Sigma a_{i_1} x_{i_1} + \cdots + \Sigma a_{i_k} x_{i_k}$, where $x_{i_t} \in S_\alpha(u_t)$, $1 \leq t \leq k$.

Define $\theta\colon R[S_\alpha] \to C_\alpha$ by $\theta(f) = \Sigma a_i(x_{i_1}-u_1) + \cdots + \Sigma a_{i_k}(x_{i_k}-u_k)$. Notice that if $x \in S_\alpha(u)$, then $\theta(x) = x-u$. That is, θ takes each generator of $R[S_\alpha]$ to one generator of

C_α. By the definition of C_α this correspondence is clearly one-to-one and onto. Hence θ is a well defined bijection.

Now let $f,g \in R[S_\alpha]$ and let $r \in R$. By definition of θ it follows that $\theta(f+g) = \theta(f)+\theta(g)$ and $\theta(rf) = r\theta(f)$. To show that θ preserves products it is sufficient to prove it for the generators. Let $x \in S_\alpha(u)$ and $y \in S_\alpha(v)$. By Lemma 4.8 (iv) $xy \in S_\alpha(uv)$; hence $\theta(xy) = xy - uv = xy-uv-uv+uv = xy-\theta_{\alpha w}(x)v-u\theta_{\alpha w}(y)+uv = (x-u)(y-v) = \theta(x)\theta(y)$. Therefore θ is an algebra isomorphism.

Theorem 4.13. $N[R[S]] = N[R[S_w]] \oplus \Sigma_{\alpha\in P_0} N[C_\alpha]$, and $J[R[S]] = J[R[S_w]] \oplus \Sigma_{\alpha\in P_0} J[C_\alpha]$, where $N[C_\alpha] \cong N[R[S_\alpha]]$ and $J[C_\alpha] \cong J[R[S_\alpha]]$.

Proof. This follows directly from Theorem 4.11 and Lemma 4.12.

Corollary 4.14. The following conditions are equivalent.

(i) $R[S_\alpha]$ is semiprime (semisimple) for each $\alpha \in P_0$.

(ii) $R[S]$ is semiprime (semisimple).

Proof. By Theorem 4.13 $N[R[S]] = 0$ $(J[R[S]] = 0)$ if and only if $N[R[S_\alpha]] = 0$ $(J[R[S_\alpha]] = 0)$ for all $\alpha \in P_0$.

Corollary 4.15. If F is a field of characteristic 0 or a prime not dividing the order of any of the groups S_α, where $|S_\alpha| < \infty$ for all $\alpha \in P_0$, then $F[S]$ is semisimple.

Proof. By Maschke's theorem [13] each $F[S_\alpha]$ is semisimple; so the result follows from Corollary 4.14.

Corollary 4.16. (1) If F is a field of characteristic 0 that is not algebraic over the rational field Q, then $F[S]$ is semisimple. (2) If F is a field of characteristic $p > 0$ that is not algebraic over the prime field $GF(p)$, and if S_α contains no elements of order p for all $\alpha \in P_0$, then $F[S]$ is semisimple.

Proof. Amitsur's theorem [22] and the hypothesis in (1), or Passman's theorem [22] and the hypothesis in (2) show that in either case $F[S_\alpha]$ is semisimple for each $\alpha \in P_0$. Hence the results follow from Corollary 4.14.

Corollary 4.17. Let F be a field. The following conditions are equivalent:

(i) $F[S]$ is regular;

(ii) for each $\alpha \in P_0$ the group S_α is locally finite and has no elements of order p in case char $F = p$.

Proof. By Theorem 4.11 and Corollary 4.12 $F[S]$ is regular if and only if $F[S_\alpha]$ is regular for all $\alpha \in P_0$. Now, Villamayor [25] and Connell [7] have shown that $F[S_\alpha]$ is regular if and only if S_α is locally finite and has no elements of order p in case char $F = p$; hence the result follows.

Corollary 4.18. Let F be a field. Assume either (1) char $F = 0$ or else (2) char $F = p > 0$ and, for each $\alpha \in P_0$, S_α has no finite normal subgroup of order divisible by p. Then the following conditions hold in $F[S]$.

(i) $F[S]$ is semiprime.

(ii) The center of $F[S]$, $Z[F[S]]$, is semiprime.

(iii) $Z[F[S]]$ is semisimple.

Moreover, if (2) happens, then these three conditions are equivalent.

Proof. From Theorem 4.13 and Lemma 4.12, it follows that $Z[F[S]]$ is semiprime (semisimple) if and only if $Z[F[S_\alpha]]$ is semiprime (semisimple) for all $\alpha \in P_0$. Hence the conclusion follows by applying Passman's results [22, Theorem 2.12 and Theorem 2.13].

Next we consider a second family of semigroups which admit relative inverses.

Definition. A semilattice P is said to be of type P_1 if the following conditions hold in P:

(i) P has a zero element;

(ii) if F is a subset of P, then either g.l.b. $\{x | x \in F\}$ belongs to F or else there exist x,y in F such that g.l.b. $\{x,y\}$ is in P-F.

Denote by τ the family of all semigroup S with the following properties: (i) S admits relative inverses; (ii) the idempotents in S commute; and (iii) S has a semilattice of subgroups of type P_1.

Let $S \in \tau$ and let $\{S_\alpha\}_{\alpha \in P}$ be the family of subgroups of S. Denote the zero element of P by α_0. Let R be a ring not necessarily with 1. Our goal now is to prove that the semigroup ring $R[S]$ is semiprime if $R[S_\alpha]$ is semiprime for each $\alpha \in P$. But first we need two preliminary results.

Lemma 4.19. There exists $\{T_i\}_{i \in I}$, a family of subsets of P, where $P = \cup_{i \in I} T_i$ and the index set I is an initial segment of the ordinals such that $T_0 \subset T_1 \subset \dots \subset T_i \subset \dots$. Moreover, if we let $S_{t_i} = \cup_{\alpha \in T_i} S_\alpha$, then $S_{t_0} \subset S_{t_1} \subset \dots \subset S_{t_i} \subset \dots$ is a series of ideals of S with $S = \cup_{i \in I} S_{t_i}$.

Proof. The sets T_i will be obtained by transfinite induction. Define $T_0 = \{\alpha_0\}$. Assume T_j is defined for all the ordinals $j < i$. If $\cup_{j<i} T_j \neq P$, let $T_i' = \{x \in P - \cup_{j<i} T_j \mid$ there exists $y \in P - \cup_{j<i} T_j$ such that g.l.b. $\{x,y\} \in \cup_{j<i} T_j\}$. If $T_i' \neq \phi$, set $T_i = T_i' \cup (\cup_{j<i} T_j)$. If $T_i' = \phi$, let $T_i'' =$ g.l.b. $\{x \mid x \in P - \cup_{j<i} T_j\}$, and set $T_i = (\cup_{j<i} T_j) \cup T_i''$. Notice that since P is of type P_1, then T_i is defined. Clearly $T_0 \subset T_1 \subset \dots \subset T_i \subset \dots$, and $P = \cup_{i \in I} T_i$. It follows that $S_{t_0} \subseteq S_{t_1} \subseteq \dots \subseteq S_{t_i} \subseteq \dots$ with $S = \cup_{i \in I} S_{t_i}$. Now let $\alpha \in T_j$ and $\beta \in T_i$, where $i,j \in I$. By construction $\alpha\beta =$ g.l.b. $\{\alpha,\beta\}$ belongs to T_k, where $k = \min\{i,j\}$. Hence $S_\alpha S_\beta \subset S_{t_k} \subseteq S_{t_j}$, and therefore $S_{t_j} S_{t_i} \subseteq S_{t_j}$. Symmetrically, $S_{t_i} S_{t_j} \subseteq S_{t_j}$. This shows that $S_{t_j} S$ and $S S_{t_j}$ are contained in S_{t_j} for all $j \in I$; that is, each S_{t_j} is an ideal of S.

Lemma 4.20. Let R be a ring not necessarily with 1. Let I be an initial segment of the ordinals, and let $\{A_i\}_{i \in I}$ be a family of ideals of R such that $A_0 \subseteq A_1 \subseteq \dots \subseteq A_i \subseteq \dots$ and $R = \Sigma_{i \in I} A_i$. Then the following conditions are equivalent.

(i) The ideals A_i are semiprime for all $i \in I$.

(ii) R is semiprime.

Proof. Assume A_i is semiprime for each $i \in I$. For contradiction, suppose that R is not semiprime, and let B be a nonzero nilpotent ideal of R. Since the ideals A_i form a chain, then $B \cap A_j \neq 0$ for some $j \in I$; so $B \cap A_j$ is a nilpotent ideal of A_j, which contradicts our hypothesis.

Conversely, assume that R is semiprime, and suppose that, for some $i \in I$, the ideal A_i is not semiprime. Let C be a nonzero nilpotent ideal of A_i. For each j,k in I, we have $CA_jCA_k \subseteq CA_jA_iA_k \subseteq CA_i \subseteq C$. Hence $(CR)^2 \subseteq C$. Thus CR is a nilpotent right ideal of R. Since R is semiprime, $CR = 0$. Thus C is a nonzero nilpotent right ideal of R, which contradicts our hypothesis. Therefore A_i is semiprime for all $i \in I$.

Theorem 4.21. Let $S \in \tau$. If $R[S_\alpha]$ is semiprime for all $\alpha \in P$, then R[S] is semiprime.

Proof. Assume $R[S_\alpha]$ is semiprime for all $\alpha \in P$. Let $\{S_{t_i}\}_{i \in I}$ be as defined in Lemma 4.19. Using transfinite induction, we show first that the ideals $R[S_{t_i}]$ are semiprime for all $i \in I$. Then by Lemma 4.20 the result follows.

The first step of the induction argument is true by hypothesis. Assume that $R[S_{t_j}]$ is semiprime for all the ordinals $j < i$, $i,j \in I$. Let $S_{t_i'} = \cup_{\alpha \in T_i'} S_\alpha$, and let

$S_{t_i''} = \cup_{\alpha\in T_i''} S_\alpha$. By the proof of Lemma 4.17 we see that either $S_{t_i} = S_{t_i'} \cup (\cup_{j<i} S_{t_j})$ or else $S_{t_i} = (\cup_{j<i} S_{t_j}) \cup S_{t_i''}$.

Suppose $S_{t_i} = (\cup_{j<i} S_{t_j}) \cup S_{t_i''}$. By definition $T_i'' = \{\alpha\}$ for some $\alpha \in P - \cup_{j<i} T_j$; hence $S_{t_i''} = S_\alpha$. Then $R[S_{t_i''}]$ is semiprime by hypothesis. By Lemma 4.19 each S_{t_j} and hence $\cup_{j<i} S_{t_j}$ is an ideal of S. It follows from the induction hypothesis and Lemma 4.20 that $R[\cup_{j<i} S_{t_j}]$ is semiprime. Let A be a nilpotent ideal of $R[S_{t_i}]$. Then $A + R[\cup_{j<i} S_{t_j}]/R[\cup_{j<i} S_{t_j}]$, being a nilpotent ideal of $R[S_{t_i}]/R[\cup_{j<i} S_{t_j}]$, is isomorphic to a nilpotent ideal of $R[S_{t_i''}]$. Since $R[S_{t_i''}]$ is semiprime, it follows that $A \subseteq R[\cup_{j<i} S_{t_j}]$. But $R[\cup_{j<i} S_{t_j}]$ is also semiprime; hence $A = 0$. Therefore $R[S_{t_i}]$ is semiprime.

Now let $S_{t_i} = (\cup_{j<i} S_{t_j}) \cup S_{t_i'}$. By definition of T_i' if $\alpha,\beta \in T_i'$, then $\alpha\beta = \text{g.l.b.}\ \{\alpha,\beta\}$ belongs to $\cup_{j<i} T_j$. Hence $S_\alpha S_\beta \subseteq \cup_{j<i} S_{t_j}$ for all $\alpha,\beta \in T_i'$. Moreover, $(\sum_{\alpha\neq\beta} R[S_\alpha]) \cap R[S_\beta] = 0$ for all β; thus we can write $R[S_{t_i'}] = \Sigma_{\beta\in T_i'} R[S_\beta]$, where all $R[S_\beta]$ are semiprime by hypothesis. As in the first case $R[\cup_{j<i} S_{t_j}]$ is a semiprime ideal of $R[S_{t_i}]$. Moreover, $R[\cup_{j<i} S_{t_j}] \cap R[S_{t_i'}] = 0$; thus we can write $R[S_{t_i}] = R[\cup_{j<i} S_{t_j}] \oplus R[S_{t_i'}] = R[\cup_{j<i} S_{t_j}] \oplus (\Sigma_{\beta\in T_i'} R[S_\beta])$ as a direct sum of R-modules. Let $f \in R[S_{t_i}]$. We may write $f = f_i + f_{\beta_1} + \ldots + f_{\beta_n}$, where $f_i \in R[\cup_{j<i} S_{t_j}]$ and $f_{\beta_i} \in R[S_{\beta_i}]$ for some $\beta_1,\ldots,\beta_n \in T_i'$.

Fix $\rho \in T_i'$ and define $\pi_\rho\colon R[S_{t_i}] \to R[S_\rho]$ by $\pi_\rho(f) = f_\rho$. Clearly π_ρ is an R-module epimorphism. To show that π_ρ is a ring epimorphism, let $f,g \in R[S_{t_i}]$ such that $f = f_i + f_\delta + f_\rho$ and $g = g_i + g_\delta + g_\rho$, where f_δ and g_δ belong to $\Sigma_{\beta\in T_i'-\{\rho\}}R[S_\beta]$. Then $fg = f_ig_i + f_i(g_\delta + g_\rho) + (f_\delta + f_\rho)g_i + f_\delta g_\rho + f_\rho g_\delta + f_\delta g_\delta + f_\rho g_\rho$. Since $R[\cup_{j<i}S_{t_j}]$ is an ideal, and since $R[S_\delta]R[S_\rho] \subseteq R[\cup_{j<i}S_{t_j}]$, then the only term in the product fg that belongs to $R[S_\rho]$ is $f_\rho g_\rho$. Hence $\pi_\rho(fg) = f_\rho g_\rho = \pi_\rho(f)\pi_\rho(g)$; so π_ρ preserves products.

Suppose $R[S_{t_i}]$ contains a nonzero nilpotent ideal B. Since $R[\cup_{j<i}S_{t_j}]$ is semiprime, then for some $\rho \in T_i'$, $\pi_\rho(B)\neq 0$. Since π_ρ is a ring epimorphism, then $\pi_\rho(B)$ is a nonzero nilpotent ideal of $R[S_\rho]$, which contradicts the hypothesis.

Corollary 4.22. Let P be an initial segment of the ordinals. Let S be a semigroup which admits relative inverses and such that every pair of its idempotents commute. Let $\{S_\alpha\}_{\alpha\in P}$ be the family of subgroups of S. Let R be a ring not necessarily with identity. If $R[S_\alpha]$ is semiprime for $\alpha \in P$, then R[S] is semiprime.

Proof. Since P is of type P_1, the result follows from Theorem 4.21.

BIBLIOGRAPHY

1. R. G. Ayoub and C. Ayoub, On the Group Ring of a Finite Abelian Group, Bull. Austral. Math. Soc. 1 (1969), 245-261.

2. B. Banaschewski, On Proving the Absence of Zero Divisors for Semigroup Rings, Canad. Math. Bull. 4 (1961), 225-232.

3. A. H. Clifford, Semigroups Admitting Relative Inverses, Ann. of Math. 42 (1941), 1049-1307.

4. A. H. Clifford and D. D. Miller, Semigroups Having Zeroid Elements, Amer. J. of Math. 70 (1948), 117-125.

5. A. H. Clifford and G. B. Preston, Algebraic Theory of Semigroups, Volume I, Math. Surveys of the Amer. Math. Soc., Rhode Island, 1961.

6. A. H. Clifford and G. B. Preston, Algebraic Theory of Semigroups, Volume II, Math. Surveys of the Amer. Math. Soc., Rhode Island, 1967.

7. I. G. Connell, Zero Divisors in Group Rings, Comm. Algebra 2 (1974), 1-14.

8. C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras. John Wiley and Sons, New York, 1962.

9. N. J. Divinsky, Rings and Radicals, Univ. of Toronto Press, 1965.

10. R. Gilmer and T. Parker, Nilpotent elements of Commutative Semigroup Rings, Michigan Math. J. 22 (1975), 97-108.

11. R. Gilmer and M. L. Teply, Units of Semigroup Rings, Comm. Algebra 5 (1977), 1275-1303.

12. G. Higman, The Units of Group Rings, Proc. London Math. Soc. 46 (1940), 231-248.

13. T. W. Hungerford, Algebra, Holt Rinehart and Winston, New York, 1974.

14. N. Jacobson, Lectures in Abstract Algebra, Volume I: Basic Concepts, Van Nostrand, New Jersey, 1951.

15. J. Lambeck, Lectures in Rings and Modules, Blaisdell, Toronto, 1966.

16. N. H. McCoy and D. Montgomery, A Representation of Generalized Boolean Rings, Duke Math. J. 3 (1937), 445-459.

17. N. H. McCoy, The Theory of Rings, Chelsea, New York, 1973.

18. W. D. Munn, On Semigroup Algebras, Proc. Cambridge Phi. Soc. 51 (1955), 1-15.

19. W. D. Munn, Matrix Representations of Inverse Semigroups, Proc. London Math. Soc. 14 (1964), 165-181.

20. T. G. Parker, Infinite Group Rings, Dissertation, Florida State University, 1973.

21. D. S. Passman, Infinite Group Rings, Marcel and Dekker, New York, 1971.

22. D. S. Passman, The Algebraic Structure of Group Rings, John Wiley and Sons, 1977.

23. F. Raggi Cardenas, Units in Group Rings, An. Inst. Mat. Univ. Nac. Autonoma Mexico 7 (1967), 27-35.

24. K. Shoda, Über die Einheitengruppe Eines Endlicher Ringes, Math. Ann. 102 (1930), 273-282.

25. O. E. Villamayor, On Weak Dimension of Algebras, Pac. J. Math. 9 (1959), 941-951.

BIOGRAPHICAL SKETCH

Antonio Quesada was born in Melilla, Spain, on January 29, 1948. He attended the University of Granada where he received the "Licenciatura" in Mathematics in June 1971. In September 1975, he enrolled at the University of Florida to continue his mathematical education.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Mark L Teply
Mark L. Teply
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Jorge Martinez
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Gerhard Ritter
Gerhard X. Ritter
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Sergio Zarantonello
Sergio E. Zarantonello
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Elroy J. Bolduc, Jr.
Associate Professor of Subject
Specialization Teacher Education

This dissertation was submitted to the Department of Mathematics in the College of Arts and Sciences, and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August 1978

Dean, Graduate School